137. *Some Boundedness Theorems of Solutions of Linear Differential Equations*

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First we shall deal with an arbitrary solution $x(t)$ of a second-order linear inhomogeneous equation

$$x''+p(t)x'+q(t)x=f(t) \tag{1}$$

where the primes denote differentiations with respect to the independent variable t and $p(t)$, $q(t)$ and $f(t)$ are assumed real-valued, continuous on the interval I: $c\leqq t<\infty$, c being a constant. We shall establish some boundedness theorems by a direct and simple method (cf. [1]). Subsequently we shall proceed to the consideration of a linear inhomogeneous system and prove an important estimate concerning the solutions of the system. The boundedness theorems for an n-th order linear inhomogeneous equation

$$x^{(n)}+p_1(t)x^{(n-1)}+\cdots+p_n(t)x=f(t) \tag{2}$$

where $p_1(t), \cdots, p_n(t)$ and $f(t)$ are all assumed real-valued, continuous on I, will then be derived as an immediate consequence of the estimate mentioned above.

THEOREM 1. *If there exists a continuous positive function $\lambda(t)$ on I such that*

$$\int_c^\infty |2p(t)+\lambda'(t)/\lambda(t)|\,dt<\infty \tag{3}$$

$$\int_c^\infty \{\lambda(t)\}^{-1/2}\,|\lambda(t)-q(t)|\,dt<\infty \tag{4}$$

$$\int_c^\infty \{\lambda(t)\}^{-1/2}\,|f(t)|\,dt<\infty, \tag{5}$$

then every solution of the equation (1) *is bounded on I. But if $2p(t)\lambda(t)+\lambda'(t)\geqq 0$ on I, then every solution of the equation* (1) *is bounded on I provided that the conditions* (4) *and* (5) *are fulfilled.*

For the proof of Theorem 1, we need the following lemma (cf. [2]).

Lemma. If $u(t)$, $v(t)\geqq 0$, if c_1 is a positive constant, and if

$$u(t)\leqq c_1+\int_c^t uv\,dt$$

then

$$u(t)\leqq c_1\exp\left(\int_c^t v\,dt\right).$$

Proof of THEOREM 1. We introduce the function

$$E(t)=1+x^2(t)+x'^2(t)/\lambda(t) \tag{6}$$

defined on I. The expression (6) may be regarded as an extension of "conjugate energy" $x^2(t)+x'^2(t)/q(t)$ associated with any solution $x(t)$ of the equation $x''+q(t)x=0$ (cf. [3]). Clearly, $E(t)>0$ on I and its first derivative becomes

$$E'(t)=-(2p\lambda+\lambda')x'^2/\lambda^2+2xx'(1-q/\lambda)+2x'f/\lambda \tag{7}$$

by virtue of the equation (1). Then we have

$$\begin{aligned}E'(t)&\leqq |2p+\lambda'/\lambda|\,x'^2/\lambda+2|xx'||1-q/\lambda|+2|x'||f|/\lambda\\ &\leqq |2p+\lambda'/\lambda|\,E(t)+(\lambda^{1/2}x^2+\lambda^{-1/2}x'^2)|1-q/\lambda|+|f|\lambda^{-1}(\lambda^{1/2}+\lambda^{-1/2}x'^2)\\ &\leqq(|2p+\lambda'/\lambda|+\lambda^{-1/2}|\lambda-q|+\lambda^{-1/2}|f|)E(t)\end{aligned}$$

and integrating both sides between c and t, we obtain

$$E(t)\leqq E(c)+\int_c^t(|2p+\lambda'/\lambda|+\lambda^{-1/2}|\lambda-q|+\lambda^{-1/2}|f|)E(t)dt.$$

Hence an application of the lemma cited above yields

$$x^2(t)<E(t)\leqq E(c)\exp\left(\int_c^t|2p+\lambda'/\lambda|dt+\int_c^t\lambda^{-1/2}|\lambda-q|dt+\int_c^t\lambda^{-1/2}|f|dt\right). \tag{8}$$

Consequently the estimate (8) and the conditions (3), (4), (5) lead at once to the conclusion that every solution $x(t)$ of the equation (1) must be bounded on I. The proof of the first part of Theorem 1 is thus completed. To prove the second part, it suffices to notice that the relation (7) gives us the inequality

$$E'(t)\leqq 2|xx'||1-q/\lambda|+2|x'||f|/\lambda$$

under the assumption $2p\lambda+\lambda'\geqq 0$ on I.

Now, putting $\lambda(t)\equiv a>0$, a being a constant, or $\lambda(t)\equiv q(t)$, we obtain the following boundedness theorems for the equation (1) as special cases of Theorem 1.

THEOREM 2. *If there exists a positive constant a such that*

$$\int_c^\infty|a-q(t)|dt<\infty \tag{9}$$

and furthermore, if the conditions

$$\int^\infty|p(t)|dt<\infty \tag{10}$$

$$\int_c^\infty|f(t)|dt<\infty \tag{11}$$

are satisfied, then every solution of the equation (1) *is bounded on I. But if $p(t)\geqq 0$ on I, then every solution of the equation* (1) *is bounded on I provided that the conditions* (9) *and* (11) *are fulfilled.*

THEOREM 3. *If $q(t)>0$ on I and if the conditions*

$$\int_c^\infty|2p(t)+q'(t)/q(t)|dt<\infty \tag{12}$$

$$\int_{0}^{\infty}\{q(t)\}^{-1/2}|f(t)|\,dt<\infty \tag{13}$$

are satisfied, then every solution of the equation (1) *is bounded on I. But if* $q(t)>0$, $2p(t)q(t)+q'(t)\geqq 0$ *on I, then every solution of the equation* (1) *is bounded on I provided that the condition* (13) *is fulfilled.*

Moreover, the following boundedness theorems concerning an equation of the type

$$(k(t)x')'+g(t)x=f(t) \tag{14}$$

can be derived as special cases of Theorems 1 and 3.

THEOREM 4. *Let both* $k(t)$ *and* $g(t)$ *be continuous functions and* $k(t)\neq 0$ *on I. If there exists a positive constant a such that*

$$\int_{0}^{\infty}\left|\frac{a}{k(t)}-g(t)\right|dt<\infty \tag{15}$$

and furthermore, if the condition (11) *is satisfied, then every solution of the equation* (14) *is bounded on I. In particular, if* $k(t)\neq 0$ *on I and if the conditions*

$$\int_{0}^{\infty}dt/|k(t)|<\infty,\quad \int_{0}^{\infty}|g(t)|\,dt<\infty,\quad \int_{0}^{\infty}|f(t)|\,dt<\infty \tag{16}$$

are satisfied, then every solution of the equation (14) *is bounded on I.*

THEOREM 5. *Let both* $k(t)$ *and* $g(t)$ *be continuous positive functions on I. If the conditions*

$$\int_{0}^{\infty}|(kg)'|/kg\,dt<\infty \tag{17}$$

$$\int_{0}^{\infty}(kg)^{-1/2}|f|\,dt<\infty \tag{18}$$

are satisfied, then every solution of the equation (14) *is bounded on I. But if* $(kg)'\geqq 0$ *on I, then every solution of the equation* (14) *is bounded on I provided that the condition* (18) *is fulfilled.*

The second part of Theorem 5 concerning the inhomogeneous equation (14) is a generalization of a theorem due to Wintner [4] and Leighton [5], which states that every solution of a self-adjoint differential equation

$$(k(t)x')'+g(t)x=0 \tag{19}$$

is bounded on I, if $k(t)$, $g(t)>0$ and if $k(t)g(t)$ is nondecreasing on I. On the other hand, it is easy to verify that the condition (17) is certainly satisfied when $k(t)g(t)$ is nonincreasing on I and has a positive lower bound, in which case the condition (18) may be replaced by the condition (11).

We shall now propose to consider a linear inhomogeneous system

$$x_i'(t)=\sum_{k=1}^{n}f_{ik}(t)x_k(t)+g_i(t)\qquad i=1,2,\cdots,n \tag{20}$$

where $f_{ik}(t)$, $i,k=1,2,\cdots,n$ and $g_i(t)$, $i=1,2,\cdots,n$ are all assumed real-valued, continuous on the interval I: $c \leqq t < \infty$, c being a constant. Here we introduce the function

$$E(t)=1+\sum_{i=1}^{n} x_i^2(t)/\lambda_i^2(t) \tag{21}$$

defined on I, where all $\lambda_i(t)$ are continuous positive functions on I. Then we can find an estimate for $E(t)$ in the following manner.

THEOREM 6. *It holds on* I:

$$\begin{aligned} E(t) \leqq E(c)\exp\Big(\sum_{i<k}^{1,n}\int_c^t |\lambda_i^{-1}\lambda_k f_{ik}+\lambda_i\lambda_k^{-1}f_{ki}|\,dt \\ +2\sum_{i=1}^{n}\int_c^t |f_{ii}-\lambda_i'\lambda_i^{-1}|\,dt+\sum_{i=1}^{n}\int_c^t \lambda_i^{-1}|g_i|\,dt\Big). \end{aligned} \tag{22}$$

Proof of THEOREM 6.

$$\begin{aligned} E'(t) &= \sum_{i=1}^{n} 2x_i x_i' \lambda_i^{-2} - 2\sum_{i=1}^{n} x_i^2 \lambda_i' \lambda_i^{-3} \\ &= \sum_{i=1}^{n} 2x_i \lambda_i^{-2}\Big(\sum_{k=1}^{n} f_{ik}x_k+g_i\Big) - 2\sum_{i=1}^{n} x_i^2\lambda_i'\lambda_i^{-3} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} f_{ik}\lambda_i^{-2}\cdot 2x_i x_k - 2\sum_{i=1}^{n} x_i^2\lambda_i'\lambda_i^{-3} + \sum_{i=1}^{n} g_i\lambda_i^{-2}\cdot 2x_i \\ &= \sum_{i<k}^{1,n}(f_{ik}\lambda_i^{-2}+f_{ki}\lambda_k^{-2})2x_i x_k + 2\sum_{i=1}^{n}(f_{ii}-\lambda_i'\lambda_i^{-1})x_i^2\lambda_i^{-2} \\ &\quad + \sum_{i=1}^{n} g_i\lambda_i^{-2}\cdot 2x_i \\ &\leqq \sum_{i<k}^{1,n}\lambda_i\lambda_k|f_{ik}\lambda_i^{-2}+f_{ki}\lambda_k^{-2}|(x_i^2\lambda_i^{-2}+x_k^2\lambda_k^{-2}) \\ &\quad + 2\sum_{i=1}^{n}|f_{ii}-\lambda_i'\lambda_i^{-1}|x_i^2\lambda_i^{-2} + \sum_{i=1}^{n}\lambda_i^{-1}|g_i|(1+x_i^2\lambda_i^{-2}) \\ &\leqq \Big(\sum_{i<k}^{1,n}|\lambda_i^{-1}\lambda_k f_{ik}+\lambda_i\lambda_k^{-1}f_{ki}| + 2\sum_{i=1}^{n}|f_{ii}-\lambda_i'\lambda_i^{-1}| + \sum_{i=1}^{n}\lambda_i^{-1}|g_i|\Big)E(t) \end{aligned}$$

and integrating both sides between c and t, we obtain

$$\begin{aligned} E(t) \leqq E(c)+\int_c^t\Big(\sum_{i<k}^{1,n}|\lambda_i^{-1}\lambda_k f_{ik}+\lambda_i\lambda_k^{-1}f_{ki}| + 2\sum_{i=1}^{n}|f_{ii}-\lambda_i'\lambda_i^{-1}| \\ + \sum_{i=1}^{n}\lambda_i^{-1}|g_i|\Big)E(t)\,dt. \end{aligned}$$

Hence an application of the lemma cited above yields the estimate (22).

One notes incidentally that the following boundedness theorem concerning the solutions of the system (20) will be derived as a consequence of the estimate (22) applied to the case: $\lambda_i \equiv 1, i=1, 2, \cdots, n$.

THEOREM 7. *All solutions* $x_i(t)$, $i=1, 2, \cdots, n$, *of the linear inhomogeneous system* (20) *are bounded on* I *provided that the conditions*

$$\int_c^\infty |f_{ik}+f_{ki}|\,dt < \infty, \quad \int_c^\infty |g_i|\,dt < \infty, \quad i, k=1, 2, \cdots, n$$

are fulfilled.

Furthermore, the n-th order linear inhomogeneous equation (2):

$$x^{(n)}+p_1(t)x^{(n-1)}+\cdots+p_n(t)x=f(t)$$

may be converted into a system of the type (20) by means of the substitutions

$$x_1=x, \quad x_2=x', \quad x_3=x'', \quad \cdots, \quad x_n=x^{(n-1)}$$

and so it is not difficult to verify that the estimate (22) for the equation (2) becomes

$$(23)\quad \begin{aligned} E(t)\leqq E(c)\exp\Big(&\sum_{i=1}^{n-2}\int_c^t \lambda_i^{-1}\lambda_{i+1}dt+\sum_{i=1}^{n-2}\int_c^t \lambda_i\lambda_n^{-1}\,|\,p_{n-i+1}\,|\,dt \\ &+\int_c^t |\,\lambda_{n-1}^{-1}\lambda_n-\lambda_{n-1}\lambda_n^{-1}p_2\,|\,dt+2\int_c^t |\,p_1+\lambda_n'\lambda_n^{-1}\,|\,dt+\int_c^t \lambda_n^{-1}\,|\,f\,|\,dt\Big)\end{aligned}$$

where

$$E(t)=1+\sum_{i=1}^{n}\{x^{(i-1)}(t)\}^2/\lambda_i^2(t).$$

Now, corresponding to particular forms of $\lambda_i(t)$, several boundedness theorems for the equation (2) will be derived as corollaries of the estimate (23). For instance, setting $\lambda_i(t)=t^{-(i-2)-n/2}$, $i=1, 2, \cdots, n$, we obtain the following theorem.

THEOREM 8. *If the conditions*

$$\int_c^\infty t^{k-1}\,|\,p_k(t)\,|\,dt<\infty, \qquad k=3, 4, \cdots, n$$

$$\int_c^\infty |\,t^{-1}-t\,p_2(t)\,|\,dt<\infty, \quad \int_c^\infty \Big|\,p_1(t)-\Big(\frac{3}{2}n-2\Big)t^{-1}\,\Big|\,dt<\infty,$$

$$\int_c^\infty t^{\frac{3}{2}n-2}\,|\,f(t)\,|\,dt<\infty$$

are satisfied for some positive constant c, then the order relations

$$x^{(m)}(t)=O(t^{-m}) \quad \text{for} \quad t\to\infty,\ m=0, 1, 2, \cdots, n-1,\ n\geqq 2,$$

hold for every solution $x(t)$ of the equation (2).

References

[1] T. Sato has also established a boundedness theorem for the equation (1); cf. T. Sato: Über Stabilität einer linearen Differentialgleichung zweiter Ordnung, Jap. Jour. Math., **10**, 195–197 (1934). However, his method is based on a theorem of Späth and the conditions given are more restrictive than ours.

[2] R. Bellman: Stability Theory of Differential Equations, Lemma 1, 35 (1953).

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[5] W. Leighton: Bounds for the solutions of a second-order linear differential equation, Proc. Nat. Acad. Sci. U. S. A., **35**, 190–191 (1949).