135. *Abstract Vanishing Cycle Theory**)

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1. *Introduction.* In this short note we shall discuss a simplified version of our abstract vanishing cycle theory[1] including the unequal-characteristic case. This theory provides, roughly speaking, abstract analogues of parabolic substitutions which the solutions of differential equations of Picard-Fuchs type undergo around the simplest type of singular points and it can be applied to construct an algebraic theory of modular functions with levels for all characteristics.[2] This we shall discuss separately[3] in the case of elliptic modular functions.

2. *Starting point.* Suppose that R is a discrete valuation ring. In order to be able to apply Hensel's lemma[4] we shall assume that R is complete. Let K be the quotient field and k the residue field. We fix a natural homomorphism of R to k and call its extensions specializations at the center of R.[5] Let C be a non-singular curve defined over K and let C' be its specialization at the center of R. We shall assume that C' is absolutely irreducible. We shall also assume that C' has at most one singularity and that the singularity is an ordinary double point. We note that ordinary singular points are, in a sense which can be made precise easily, generic singularities. At any rate, we shall denote this possible singular point by Q. If g is the genus of C, the genus of C' is either g or $g-1$ according as Q is absent or not. Pick a divisor $\mathfrak{r}$ of C of degree d greater than $2g-2$ rational over K such that the specialization $\mathfrak{r}'$ at the center of R is free from Q. This is always possible and, in fact, we can even assume that $\mathfrak{r}$ is positive. Let J be the Jacobian variety of C constructed by Chow's method[6] with reference to $\mathfrak{r}$. Then the specialization J' of J at the center of R is either the Jacobian variety of C' constructed by Chow's method or a completion of the Rosenlicht variety $(J')_0$ of C' constructed by Chow's method[7] with reference to $\mathfrak{r}'$. Moreover, the image points of $\mathfrak{r}$ and $\mathfrak{r}'$ being taken as neutral elements of J and $(J')_0$, the group law of J is specialized to the group law of $(J')_0$ at the center of R. We proved this *compatibility* only in the geometric case.[8] However the proof can be taken over verbatim to the present case. We also note that the Rosenlicht variety $(J')_0$ is a commutative group variety which contains the group variety G_m of

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the multiplicative group of the universal domain over k as a subgroup with the Jacobian variety of C' as the corresponding factor group. We are assuming here that C' does have a singular point. It might be unnecessary to remind that J is defined over K while $(J')_0$ and G_m are defined over k.

3. *Invariant and vanishing points.* Let n be a natural number not divisible by the characteristic of k. Let Ω be the group of points of order n on J. Then $K(\Omega)$ is a finite separable normal, i.e. a finite Galois extension of K not trivial in general. Similarly, if Ω' is the group of points of $(J')_0$ of order n, then $k(\Omega')$ is a finite Galois extension of k. Moreover, in the specialization of Ω at the center of R every member of Ω' appears with multiplicity one. The reason for this is the same as in the geometric case: If we consider the graph Γ in the product $J\times J$ of the endomorphism $u\rightarrow n\cdot u$ of J, the specialization Γ' of Γ at the center of R contains the closure of the graph $(\Gamma')_0$ in the product $(J')_0\times(J')_0$ of the endomorphism $u'\rightarrow n\cdot u'$ of $(J')_0$ as a simple component. Moreover, if we project other components of Γ' to the first factor of the product, we get a subset of the singular locus of J'. Thus the positivity and the unicity of the multiplicity of every member of Ω' in the specialization of Ω at the center of R follows from the intersection-theory. Therefore Ω contains a subgroup Ω_i which is specialized *isomorphically* onto Ω' at the center of R. According to Hensel's lemma, the group Ω_i is uniquely determined and $K(\Omega_i)$ is an unramified finite Galois extension of K. In case C' is non-singular, i.e., in case J' is the Jacobian variety of C', we have $\Omega_i=\Omega$, hence $K(\Omega)$ is unramified over K. If we exclude this trivial case, then Ω' contains a cyclic subgroup of order n which comes from G_m. Therefore Ω_i contains a subgroup Ω_v which is specialized isomorphically onto that cyclic group at the center of R. This Ω_v is also uniquely determined and we call Ω_v the *group of vanishing points of order n*. The set-theoretic complement of Ω_i in Ω is the set of "non-invariant points" of order n. We note that our terminology comes from the Lefschetz vanishing cycle theory.[9] In fact vanishing points of order n are obtained by the n-th division of period along vanishing cycle while invariant points of order n are obtained by the n-th division of periods along locally invariant $2g-1$ cycles.

4. *A pairing theorem.* Assume in general that K is an arbitrary field. We assume that n is a natural number not divisible by the characteristic of K and Ω is the group of points of J of order n. Following Weil, to each pair (s, t) of elements of Ω we can associate an n-th root of unity $e(s, t)$ so that we get a skew-symmetric pairing of Ω to itself.[10] The definition implies that $e(s, t)$ is contained in $K(s, t)$. In fact, let M be a generic point of C over $K(\Omega)$ and let φ

be the canonical function of C normalized by $\varphi(M)=0$. Then φ is defined over $K(M)$, hence over $K(\Omega, M)$. Let $M_1, \cdots, M_{g-1}$ be independent generic points of C over $K(\Omega, M)$ and let Θ be the locus of the point $\sum_{i=1}^{g-1}\varphi(M_i)$ of J over $K(\Omega, M)$. Then $e_{\Theta,n}(s,t)=e(s,t)$ is contained in $K(s,t,M)$. However, since $K(s,t,M)$ is regular over $K(s,t)$, we see that $e(s,t)$ is contained in $K(s,t)$ as asserted. *Therefore $K(\Omega)$ always contains the field of n-th roots of unity.* On the other hand, if σ is an automorphism of $K(\Omega)$ over K, the definition of $e(s,t)$ implies $e(\sigma s, \sigma t)=\sigma e(s,t)$. We know that Ω is a vector space of dimension $2g$ over integers modulo n while the multiplicative group of n-th roots of unity is a vector space of dimension one over integers modulo n. *Therefore the automorphism σ induces linear transformations $M(\sigma)$ and $m(\sigma)$ of these vector spaces and the above relation implies*

$$\det. M(\sigma)\equiv m(\sigma)^g \mod n.$$

In particular, if K contains the field of n-th roots of unity, the linear transformation $M(\sigma)$ is unimodular in the sense $\det. M(\sigma)\equiv 1 \mod n$. The proof is not quite trivial, but, if we make use of the connectedness of the symplectic group,[11] it is immediate. The above remarks will play a rôle in our later papers. Now we shall assume again that K is complete with respect to a real discrete valuation and we shall prove the following theorem:

THEOREM 1. *The two groups Ω_i and Ω_v are the groups of annihilators of each other in Ω* (with respect to the skew-symmetric pairing).

This theorem can be proved directly by examining the specialization of the theta divisor Θ. However, even in the geometric case, the proof along this line is not simple. A shorter proof can be obtained, as in the geometric case, by using another definition of $e(s,t)$, which is as follows: Let $\mathfrak{a}$ and $\mathfrak{b}$ be two divisors of C of degree zero representing s and t. Then $n\cdot\mathfrak{a}$ and $n\cdot\mathfrak{b}$ are divisors of functions f and h on C. If $\mathfrak{a}$ and $\mathfrak{b}$ are taken to have no point in common, we have

$$e(s,t)=h(\mathfrak{a}):f(\mathfrak{b}).$$ [12]

Now, if s and t are elements of Ω_i, they are specialized to simple points s' and t' of J' over any specialization of Ω_i at the center of R. Let $e(s,t)'$ be the specialization of $e(s,t)$ over the specialization $(s,t)\to(s',t')$ at the center of R. If we pick $\mathfrak{a}$ and $\mathfrak{b}$ suitably, in the specialization $(\mathfrak{a}',\mathfrak{b}')$ of $(\mathfrak{a},\mathfrak{b})$ over the specialization $(s,t,e(s,t))\to(s',t',e(s,t)')$ at the center of R both $\mathfrak{a}'$ and $\mathfrak{b}'$ come to be free from Q and have no point in common. The construction is similar as in the geometric case, hence we shall not go into detail. Consider the non-singular model C^* of C'. Let $\mathfrak{a}^*$ and $\mathfrak{b}^*$ be the unique transforms of $\mathfrak{a}'$ and $\mathfrak{b}'$ on C^*. Then $n\cdot\mathfrak{a}^*$ and $n\cdot\mathfrak{b}^*$ are divisors of functions f^*

and h^* on C^* and we have

$$e(s,t)'=h^*(\mathfrak{a}^*):f^*(\mathfrak{b}^*).$$

However, if t belongs not only to Ω_i but also to Ω_v, then $\mathfrak{b}^*$ itself is a divisor of a function h^{**} on C^* and we can assume that h^* is just the n-th power of h^{**}. This implies $e(s,t)'=1$. Since n is not divisible by the characteristic of k, we get $e(s,t)=1$. We note that Ω_i is a direct product of $2g-1$ cyclic groups of order n while Ω_v is a cyclic group of order n. Since the whole group Ω is the direct product of $2g$ cyclic groups of order n, we see that Ω_i and Ω_v are mutually the groups of all annihilators. This proves the theorem.

5. *Parabolic substitutions.* Now we shall apply the pairing theorem to determine how the inertia group of $K(\Omega)$ over K operates on Ω. The result can be stated as follows:

THEOREM 2. *Suppose that $K(\Omega_i)$ contains the field of n-th roots of unity. Then an element s of Ω and its conjugate s' over $K(\Omega_i)$ differ only by an element of Ω_v.*

Let t be an arbitrary element of Ω_i. Then by definition $e(s',t)$ is the conjugate of $e(s,t)$ over $K(\Omega_i)$, whence $e(s',t)$ coincides with $e(s,t)$. This implies $e(s'-s,t)=1$ for all t in Ω_i, hence by the pairing theorem $s'-s$ is an element of Ω_v. This is what we wanted to prove.

As a consequence $K(\Omega)$ is tamely ramified over K. In order to make the content of Theorem 2 much clearer, assume that k is algebraically closed. Then we have $K(\Omega_i)=K$ and K contains the field of n-th roots of unity. Therefore, if we take a base of Ω so that the second axis is along Ω_v while the second up to the last axes are along Ω_i, the Galois group of $K(\Omega)$ over K operates on Ω as follows:

$$\begin{pmatrix} 1 & m & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \mod n.$$

In particular the Galois group of $K(\Omega)$ over K is isomorphic to a subgroup of the additive group of integers modulo n.

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