# 131. A Linear Representation of a Countably Infinite Group

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1958)

1. Let $\mathcal{G}$ be a countably infinite group and $\mathcal{H}$ the Hilbert space of all complex-valued functions $g \to f(g)$ such that $\sum_{g \in \mathcal{G}} |f(g)|^2$ is finite. For each $g \in \mathcal{G}$, let $U_g$ be the unitary operator on $\mathcal{H}$ defined by $[U_g f](g') = f(g'g)$ and let $M(\mathcal{G})$ the ring of operators generated by $\{U_g\}_{g \in \mathcal{G}}$. Murray and von Neumann have shown that $M(\mathcal{G})$ is a factor of type $\mathrm{II}_1$ if all non-trivial conjugate classes of $\mathcal{G}$ are infinite, and further proposed to expand an arbitrary countably infinite group to a group which has the above property. These results can also be interpreted in the following way: An arbitrary countably infinite group admits a faithful representation on a group of inner automorphisms of a factor of the case ($\mathrm{II}_1$) on a separable Hilbert space.

The object of the present paper is to show the following

**Theorem.** *Let G be an arbitrary countable group, then G is isomorphic to a group of outer automorphisms of the approximately finite factor on a separable Hilbert space.*

By an automorphism of a factor, we understand a $*$-automorphism, and by a group of outer automorphisms of a factor, we understand a group of automorphisms in which all but the unit element are outer. In proving our theorem, it is sufficient to show the case where $G$ is countably infinite. Indeed, let $G$ be a finite group. Then, for any countably infinite group $G'$ (for example the additive group of integers), the direct product $G \times G'$ is countably infinite and $G$ is embedded isomorphically into $G \times G'$.

The restriction that $G$ is countably infinite is not essential. For an arbitrary group, such a representation will probably be possible, because it will probably be represented as a group of outer automorphisms of a generalized approximately finite factor on an arbitrary (not necessarily separable) Hilbert space. Only for the sake of the simplicity, we confine a group $G$ to be countable.

Noting that approximately finite factors on a separable Hilbert space are all $*$-isomorphic to each other [2], our theorem yields that *an approximately finite factor on a separable Hilbert space has a group of outer automorphisms isomorphic to an arbitrary countable group.* Actually, this note arose from the investigation of the crossed products of rings of operators.[1]

1) Cf. N. Suzuki: Crossed products of rings of operators, to appear.

2. Let $G$ be an arbitrary countably infinite group and let $\varDelta$ the set of all functions $\alpha(g)$ on $G$ as follows: $\alpha(g)=1$ on a finite subset of $G$, and $=0$ otherwise. Define the addition in $\varDelta$: for $\alpha(g)$, $\beta(g)\in\varDelta$, $[\alpha+\beta](g)=\alpha(g)+\beta(g)$ (mod 2), then $\varDelta$ is obviously an additive group with the unit $0(g)=0$ $(g\in G)$. Let $\varDelta'$ be the set of all functions $\varphi(\gamma)$ on $\varDelta$ as follows: $\varphi(\gamma)=1$ on a finite subset of $\varDelta$, and $=0$ otherwise, and make $\varDelta'$ into an additive group by defining the addition: for $\varphi(\gamma)$, $\psi(\gamma)\in\varDelta'$, $[\varphi+\psi](\gamma)=\varphi(\gamma)+\psi(\gamma)$ (mod 2). Now we define the operation on $\varDelta'$ as follows: for $\varphi\in\varDelta'$, $\alpha\in\varDelta$,

$$\varphi^{\alpha}(\gamma)=\varphi(\gamma+\alpha),$$

and make the pair $(\varDelta',\varDelta)$ into a group by defining

$$(\varphi,\alpha)(\psi,\beta)=(\varphi^{\beta}+\psi,\alpha+\beta)$$

for $\alpha,\beta\in\varDelta,\varphi,\psi\in\varDelta'$. Then the unit of the group $\mathcal{G}=(\varDelta',\varDelta)$ is $(0,0)$, 0 being the unit of $\varDelta'$ and $\varDelta$ with the same notation, and the inverse is clearly $(\varphi,\alpha)^{-1}=(\varphi^{-\alpha},-\alpha)=(\varphi^{\alpha},\alpha)$. Let $H$ be the Hilbert space of all complex-valued functions $(\varphi,\alpha)\to f((\varphi,\alpha))$ such that $\sum_{(\varphi,\alpha)\in\mathcal{G}}|f((\varphi,\alpha))|^2$ is finite, and for each $(\varphi,\alpha)\in\mathcal{G}$ let $V_{(\varphi,\alpha)}$ be a unitary operator on $H$ defined by $[V_{(\varphi,\alpha)}f]((\psi,\beta))=f((\psi,\beta)(\varphi,\alpha))$. We denote by $M$ the ring of operators generated by $\{V_{(\varphi,\alpha)}\}_{(\varphi,\alpha)\in\mathcal{G}}$.

At first, it must be shown that $M$ is a factor of type $\mathrm{II}_1$. Indeed, assume that $(\varphi,\alpha)\neq(0,0)$. If $\varphi\neq 0$ then $(0,\beta)^{-1}(\varphi,\alpha)(0,\beta)=(\varphi^{\beta},\alpha)$ yields that $(0,\beta)^{-1}(\varphi,\alpha)(0,\beta)$ are infinitely many since there are infinite many distinct $\varphi^{\beta}(\beta\in\varDelta)$.[2] If $\varphi=0$ then $(\psi,0)^{-1}(0,\alpha)(\psi,0)=(\psi^{\alpha}+\psi,\alpha)$ and there exist infinitely many distinct $\psi^{\alpha}+\psi$ if $\psi$ runs over $\varDelta'$.[3] Therefore, all non-trivial conjugate classes are infinite. We can conclude that $M$ is a factor of type $\mathrm{II}_1$.

Moreover, we obtain the following

**Lemma 1.** *The group $\mathcal{G}$ is locally finite and $M$ is an approximately finite factor.*

**Proof.** First we see easily that $\varDelta$ is locally finite. Let $\mathcal{F}$ be a finite set of $\mathcal{G}$, then each of the sets $\varDelta'_{\mathcal{F}}=\{\varphi;(\varphi,\alpha)\in\mathcal{F}\}$, $\varDelta_{\mathcal{F}}=\{\alpha;(\varphi,\alpha)\in\mathcal{F}\}$ and $\varDelta_{\mathcal{F}_0}=\bigcup_{\varphi\in\varDelta'_{\mathcal{F}}}\{\alpha\in\varDelta;\varphi(\alpha)=1\}$ is a finite set. Denoting by $\bar{\varDelta}_{\mathcal{F}}$ the finite subgroup of $\varDelta$ generated by the finite set $\varDelta_{\mathcal{F}}\cup\varDelta_{\mathcal{F}_0}$ and setting $\bar{\varDelta}'_{\mathcal{F}}=\{\varphi\in\varDelta';\varphi(\alpha)=0$ on the outside of $\bar{\varDelta}_{\mathcal{F}}\}$, it is easily verified that $(\bar{\varDelta}'_{\mathcal{F}},\bar{\varDelta}_{\mathcal{F}})$ is a finite group containing $\mathcal{F}$. Hence $\mathcal{G}$ is locally finite and

2) Since $\varphi\neq 0$, there is an $\alpha_0\in\varDelta$ such that $\varphi(\alpha_0)=1$. If we pick up a sequence $\{\beta_i\}$ in $\varDelta$ such that $\alpha_0+\beta_i$ are all distinct, then each $\varphi^{\beta_i}$ takes the value 1 on $\alpha_0+\beta_i$. Since each $\varphi^{\beta_i}$ takes the value 1 on a finite set of $\varDelta$, there must be an infinite number of distinct $\varphi^{\beta_i}$.

3) Let $\varphi_i$ be a sequence in $\varDelta'$ such that $\psi_i(\beta_i)=1$, and $=0$ otherwise, where $\{\beta_i\}$ is a sequence of all distinct elements in $\varDelta$. Then $\psi_i^{\alpha}+\psi_i=\psi_j^{\alpha}+\psi_j$ implies $\beta_i+\alpha=\alpha_j$ and $\beta_j+\alpha=\beta_i$. Thus there can never coincide more than two $\psi_i^{\alpha}+\psi_i$, and hence there are infinitely many different $\psi_i^{\alpha}+\psi_i$.

we see by [2, Lemma 5.6.1] that $M$ is an approximately finite factor.

3. Put $\mathcal{G}_0=(0, \varDelta)$, then $\mathcal{G}_0$ is an abelian subgroup of $\mathcal{G}$ and the subring $M_0$ in $M$ corresponding to $\mathcal{G}_0$ is also abelian.

We shall prove the following

**Lemma 2.** *$M_0$ is a maximal abelian subring of $M$ which possesses the property:*

( * ) *A unitary operator $U$ of $M$ such that $U^{-1}M_0U\subseteqq M_0$ belongs to $M_0$.*

In order to prove this lemma, we need the following lemma in [1].

**Lemma 3.** *Assume that* (I) *for each $g\in\mathcal{G}$, $g\bar{\in}\mathcal{G}_0$, the set $\{g_0gg_0^{-1};\ g_0\in\mathcal{G}_0\}$ is infinite, and* (II) *for each finite set $\mathcal{F}$ of $\mathcal{G}$, there exists $g_1\in\mathcal{G}_0$ such that*

(1) *for each $g\in\mathcal{F}$, $g^{-1}g_1g\in\mathcal{G}_0$ implies $g\in\mathcal{G}_0$,*

(2) *the conditions $g, g'\in\mathcal{F}$, $g^{-1}g_1g'=g_1$ imply $g=g'$. Then $M_0$ is a maximal abelian subring which possesses the property* (*).

**Proof of Lemma 2.** We have seen in the preceding section that $\mathcal{G}_0$ fulfils the property (I) in Lemma 3. Thus it needs only to prove that $\mathcal{G}_0$ fulfils the property (II). Put $\varDelta_0'=\{\varphi; (\varphi, \alpha)\in\mathcal{F}\}$ for each finite set $\mathcal{F}$ of $\mathcal{G}$, $\varDelta_0'$ is finite. Setting

$$\varDelta_0=\bigcup_{\varphi\in\varDelta_0'}\{\gamma\in\varDelta; \varphi(\gamma)=1\},$$

$\varDelta_0$ is also finite, and hence the set $\varDelta_0+\varDelta_0$ is finite. Since $\varDelta$ is infinite, there exists an $\alpha_0\in\varDelta$ such that $\alpha_0\bar{\in}\varDelta_0+\varDelta_0$. Then

$$(\varphi, \alpha)^{-1}(0, \alpha_0)(\psi, \beta)=(\varphi^{\alpha}, \alpha)(0, \alpha_0)(\psi, \beta)=(\varphi^{\alpha+\alpha_0+\beta}+\psi, \alpha+\alpha_0+\beta).$$

Ad (1). For $(\varphi, \alpha)\in\mathcal{F}$, $(\varphi, \alpha)^{-1}(0, \alpha_0)(\varphi, \alpha)=(\varphi^{\alpha_0}+\varphi, \alpha_0)\in\mathcal{G}_0$ implies $\varphi^{\alpha_0}+\varphi=0$, or $\varphi^{\alpha_0}=\varphi$. If $\varphi\neq0$, $\varphi(\gamma)\not\equiv0$ on $\varDelta_0$. On the other hand, since $(\varDelta_0+\alpha_0)\frown\varDelta_0=\phi$, $\varphi^{\alpha_0}(\gamma)=\varphi(\gamma+\alpha_0)\equiv0$ on $\varDelta_0$. This contradiction yields $\varphi=0$, or $(\varphi, \alpha)\in\mathcal{G}_0$.

Ad (2). For $(\varphi, \alpha)$, $(\psi, \beta)\in\mathcal{F}$, $(\varphi, \alpha)^{-1}(0, \alpha_0)(\psi, \beta)=(0, \alpha_0)$ implies $\alpha=\beta$ and $\varphi^{\alpha_0}=\psi$. If $\psi\neq0$, $\psi(\gamma)\not\equiv0$ on $\varDelta_0$, but as seen in above, $\varphi^{\alpha_0}(\gamma)\equiv0$ on $\varDelta_0$. This contradiction yields $\varphi=\psi=0$, or $(\varphi, \alpha)=(\psi, \beta)$.

4. In the sequel, we shall consider to represent $G$ on a group of automorphisms of $M$.

For this purpose, it is necessary to map $G$ on a group of automorphisms of $\varDelta$ and $\varDelta'$.

**Lemma 4.** *For each $g\in G$, define a transformation $T_g$ on $\varDelta$ as follows:*

$$[T_g\alpha](g')=\alpha(gg') \text{ for all } \alpha\in\varDelta,$$

*and further define a transformation $T_g'$ on $\varDelta'$ as follows:*

$$[T_g'\varphi](\alpha)=\varphi(T_{g^{-1}}\alpha) \text{ for all } \varphi\in\varDelta'.$$

*Then the mapping $g\to T_g(T_g')$ is an anti-isomorphism of $G$ onto a group of automorphisms of $\varDelta(\varDelta')$ respectively.*

**Proof.** It is clear that $T_g(g \in G)$ are automorphisms of $\Delta$. For all $\alpha \in \Delta$,

$$[T_{g_1}T_{g_2}\alpha](g')=[T_{g_2}\alpha](g_1g')=\alpha(g_2g_1g')=[T_{g_2g_1}\alpha](g'),$$

hence $g \to T_g$ is an anti-homomorphism of $G$ onto a group of automorphisms $\{T_g\}$ of $\Delta$. It must be shown that it is an anti-isomorphisms. Indeed, if $g \neq e$, for a fixed $g_0 \in G$, there is an $\alpha_0 \in \Delta$ such that $\alpha_0(g_0)=0$ and $\alpha_0(gg_0)=1$, and so $T_g\alpha_0 \neq \alpha_0$.

For all $\varphi \in \Delta'$,

$$[T'_{g_2}T'_{g_1}\varphi](\alpha)=[T'_{g_1}\varphi](T_{g_2^{-1}}\alpha)=\varphi(T_{g_1^{-1}}T_{g_2^{-1}}\alpha)$$
$$=\varphi(T_{(g_1g_2)^{-1}}\alpha)=[T'_{g_1g_2}\varphi](\alpha)$$

implies $T'_{g_1g_2}=T'_{g_2}T'_{g_1}$. Hence, for the remainder, the similar one to the above proof is adapted.

**Lemma 5.** *For each $g \in G$, define the operator $U_g$ on $H$ as follows:*

$$[U_gf]((\varphi, \alpha))=f((T'_g\varphi, T_g\alpha)) \text{ for all } f \in H.$$

*Then the mapping $g \to U_g$ is a faithful unitary representation of $G$ on $H$.*

**Proof.** Each $U_g$ is unitary: For each $f \in H$,

$$\| U_gf \|^2=\sum_{(\varphi, \alpha)\in \mathfrak{G}} | [U_gf]((\varphi, \alpha)) |^2=\sum_{(\varphi, \alpha)\in \mathfrak{G}} |f((T'_g\varphi, T_g\alpha))|^2$$
$$=\sum_{(\psi, \beta)\in \mathfrak{G}} |f((\psi, \beta))|^2=\|f\|^2,$$

and so $U_gf \in H$ and $U_g$ is unitary. Further the remainder of the proof is assured by Lemmas 4, 5.

**Lemma 6.** *For each $g \in G$, define a mapping $\theta_g$ of $M$ as follows:*

$$V^{\theta_g}_{(\varphi, \alpha)}=U_{g^{-1}}V_{(\varphi, \alpha)}U_g \text{ for all } V_{(\varphi, \alpha)} \in M.$$

*Then the mapping $g \to \theta_g$ is a faithful representation of $G$ onto a group of automorphisms of $M$.*

**Proof.** By the above lemma, we need only to prove that each $\theta_g$ is an automorphism of $M$. Indeed, first noting that

$T'_g(T'_{g^{-1}}\psi)^\alpha(\gamma)=(T'_{g^{-1}}\psi)^\alpha(T_{g^{-1}}\gamma)=[T'_{g^{-1}}\psi](T_{g^{-1}}\gamma+\alpha)=\psi(T_gT_{g^{-1}}\gamma+T_g\alpha)$ $=\psi(\gamma+T_g\alpha)=\psi^{T_g\alpha}(\gamma)$ for $\psi \in \Delta'$ and $\alpha \in \Delta$, we obtain that for each $f \in H$,

$$[U_{g^{-1}}V_{(\varphi, \alpha)}U_gf]((\psi, \beta))=[V_{(\varphi, \alpha)}U_gf]((T'_{g^{-1}}\psi, T_{g^{-1}}\beta))$$
$$=[U_gf]((T'_{g^{-1}}\psi, T_{g^{-1}}\beta)(\varphi, \alpha))=[U_gf](((T'_{g^{-1}}\psi)^\alpha+\varphi, T_{g^{-1}}\beta+\alpha))$$
$$=f((T'_g(T'_{g^{-1}}\psi)^\alpha+T'_g\varphi, \beta+T_g\alpha))=f((\psi^{T_g\alpha}+T'_g\varphi, \beta+T_g\alpha))$$
$$=f((\psi, \beta)(T'_g\varphi, T_g\alpha))=[V_{(T'_g\varphi, T_g\alpha)}f]((\psi, \beta)).$$

Hence $U_{g^{-1}}V_{(\varphi, \alpha)}U_g=V_{(T'_g\varphi, T_g\alpha)}$.

Now the theorem is readily followed from the above lemmas.

**The proof of the theorem.** By Lemma 6, we see that each $\theta_g$ is an automorphism of $M$ conserving $M_0$, and hence it is sufficient from Lemma 2 to prove that each $\theta_g$ does not keep $M_0$ elementwise invariant. In fact,

$$V_{(0, \alpha)}=U_{g^{-1}}V_{(0, \alpha)}U_g=V_{(0, T_g\alpha)}$$

implies $(0, \alpha)=(0, T_g\alpha)$, or $\alpha=T_g\alpha$ for all $\alpha \in \Delta$, and hence $g=e$.

## References

[1] J. Dixmier: Sous-anneaux abéliens maximaux dans les facteurs de type fini, Ann. Math., **59**, 279–286 (1954).

[2] F. J. Murray and J. von Neumann: On rings of operators IV, Ann. Math., **44**, 716–808 (1943).