2. Notes on Tauberian Theorems for Riemann Summability. II

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In this note we shall deal with the problem proposed in §12 of Yano [6]. We prove a theorem (Theorem 1) concerning Riemann summability by using Lemma 3. Riemann summability of $\sum a_n$ is closely connected with Cesàro summability of an even function $\varphi(t)\in L$ with Fourier coefficients a_n. Here we notice that in Riemann summability a_n are independent of Fourier coefficients. Lemma 1 will interpret the relation between these two summabilities by the help of Lemmas 2 and 4; — this is a chief object of this paper. In §3 we shall give "Riemann-Cesàro summability"—analogue.

1. Riemann summability. A series

$$\sum a_\nu = \sum_{\nu=1}^{\infty} a_\nu \quad (a_0=0)$$

is said to be summable to sum s by Riemann method of order p, or briefly summable (R, p) to s, if the series in

$$F(t)=\sum_{\nu=1}^{\infty} a_\nu\left(\frac{\sin \nu t}{\nu t}\right)^p$$

converges in some interval $0<t<t_0$, and $F(t)\to s$ as $t\to 0$ (cf. Verblunsky [1]). Here we suppose that p is a positive integer, and a_n are real throughout this paper.

The n-th Cesàro sum of order r of $\sum a_\nu$ is

$$s_n^r = \sum_{\nu=0}^{n} A_{n-\nu}^r a_\nu \qquad (-\infty<r<\infty),$$

where A_n^r is defined by the identity

$$(1-x)^{-r-1}=\sum_{n=0}^{\infty} A_n^r x^n \qquad (|x|<1),$$

and in particular $a_n=s_n^{-1}$.

THEOREM 1. Let $-1\leqq b$,*) $b<p-1<\gamma<\beta$, and $\delta=\dfrac{p-1-b}{\beta-p+1}(\beta-\gamma)$. If

$$\sum_{\nu=1}^{n} |s_\nu^\beta| = o(n^{\gamma+1}) \tag{1.1}$$

$$\sum_{\nu=n}^{2n} (|s_\nu^b|-s_\nu^b)=O(n^{b+\delta+1}) \tag{1.2}$$

as $n\to\infty$, then $\sum a_\nu$ is summable (R, p) to zero.

In the case $b=-1$ we have the following corollary.

*) We could remove the restriction $b\geqq -1$ in this theorem by the argument used in Yano [5].

COROLLARY 1. Let $p-1<\gamma<\beta$ and $\delta=p(\beta-\gamma)/(\beta-p+1)$. If (1.1) holds and

$$\sum_{\nu=n}^{2n}(|a_\nu|-a_\nu)=O(n^\delta), \tag{1.2$'$}$$

then $\sum a_\nu$ is summable (R, p) to zero.

This is a theorem due to Kanno [2] when (1.2)$'$ is replaced by $\sum_{\nu=n}^{2n}|a_\nu|=O(n^\delta)$, and δ is so restricted as $0<\delta<1$.

2. Preliminary lemmas

LEMMA 1. For a series $\sum a_\nu$ to be summable (R, p) to sum s, it is sufficient that

$$\frac{1}{t^p}\sum_{\nu=1}^{\infty}a_\nu\int_0^t(t-u)^{p-1}\cos\nu u\,du\to\frac{s}{p} \qquad (t\to 0). \tag{2.1}$$

Inversely, the condition (2.1) is necessary when $p\leqq 2$.

Proof. From Hobson [7, p. 281], we have

$$(\sin t)^p=(-1)^{p/2}\left(\frac{1}{2}\right)^{p-1}\sum_{\mu=0}^{p/2-1}(-1)^\mu\binom{p}{\mu}\cos(p-2\mu)t+\left(\frac{1}{2}\right)^p\binom{p}{p/2} \quad (p,\text{ even})$$

$$=(-1)^{(p-1)/2}\left(\frac{1}{2}\right)^{p-1}\sum_{\mu=0}^{(p-1)/2}(-1)^\mu\binom{p}{\mu}\sin(p-2\mu)t \qquad (p,\text{ odd}).$$

Replacing t by nt, differentiating with respect to t p-times, and then dividing both sides by n^p we get

$$\left(\frac{d}{dt}\right)^p\left(\frac{\sin nt}{n}\right)^p=\left(\frac{1}{2}\right)^{p-1}\sum_{\mu=0}^{[(p-1)/2]}(-1)^\mu\binom{p}{\mu}(p-2\mu)^p\cos(p-2\mu)nt, \tag{2.2}$$

in the unified form. On the other hand, clearly

$$\left(\frac{\sin nt}{nt}\right)^p=\frac{1}{\Gamma(p)}\frac{1}{t^p}\int_0^t(t-u)^{p-1}\left(\frac{d}{du}\right)^p\left(\frac{\sin nu}{n}\right)^p du. \tag{2.3}$$

Substituting (2.2) into the integrand of (2.3) we have

$$\left(\frac{\sin nt}{nt}\right)^p=\left(\frac{1}{2}\right)^{p-1}\frac{1}{\Gamma(p)}\sum_{\mu=0}^{[(p-1)/2]}(-1)^\mu\binom{p}{\mu}(p-2\mu)^p\cdot \cdot\frac{1}{t^p}\int_0^t(t-u)^{p-1}\cos(p-2\mu)nu\,du. \tag{2.4}$$

Tending t to zero in both sides of (2.4) with $n=1$, we have the identity

$$1=\left(\frac{1}{2}\right)^{p-1}\frac{1}{\Gamma(p)}\sum_{\mu=0}^{[(p-1)/2]}(-1)^\mu\binom{p}{\mu}(p-2\mu)^p\frac{1}{p}. \tag{2.5}$$

Now, writing $t_\mu=(p-2\mu)t$, (2.4) becomes

$$\left(\frac{\sin nt}{nt}\right)^p=\left(\frac{1}{2}\right)^{p-1}\frac{1}{\Gamma(p)}\sum_{\mu=0}^{[(p-1)/2]}(-1)^\mu\binom{p}{\mu}(p-2\mu)^p\cdot \cdot\frac{1}{t_\mu^p}\int_0^{t_\mu}(t_\mu-u)^{p-1}\cos nu\,du.$$

Hence, if for each $\mu=0,1,\cdots,[(p-1)/2]$

$$\frac{1}{t_\mu^p}\sum_{\nu=1}^{\infty}a_\nu\int_0^{t_\mu}(t_\mu-u)^{p-1}\cos\nu u\,du\to\frac{s}{p} \qquad (t_\mu\to 0),$$

which is (2.1), then we have

$$\sum_{\nu=1}^{\infty} a_\nu \left(\frac{\sin \nu t}{\nu t}\right)^p \to \left(\frac{1}{2}\right)^{p-1} \frac{1}{\Gamma(p)} \sum_{\mu=0}^{[(p-1)/2]} (-1)^\mu \binom{p}{\mu} (p-2\mu)^p \frac{s}{p} \qquad (t\to 0).$$

And the right hand side is s by (2.5). This proves the sufficiency.

The necessity for the case $p\leqq 2$ is evident by the identity (2.4), since then its right hand side contains one term only. Thus we get the lemma.

LEMMA 2. Let $r>0$, $q\gtreqless 0$ be arbitrary, and let k be an integer such as $k>\sup(1, r-q)$. Then

$$\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{r-1} u^k \cos \nu u \, du = o(t^{q+k}) \qquad (t\to 0) \tag{2.6}$$

implies

$$\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{r-1} \cos \nu u \, du = o(t^q) \qquad (t\to 0), \tag{2.7}$$

provided that the series in (2.6) converges uniformly in every interval $0<\eta\leqq t\leqq \pi$.

Of course this lemma holds when $0<\eta\leqq t\leqq \pi$ is replaced by $0\leqq t\leqq \pi$.

For the proof we need a lemma.

LEMMA 2.1. Let $r>0$, $q\gtreqless 0$ be arbitrary, and let k be an integer such as $k>\sup(1, r-q)$. Then a necessary and sufficient condition for

$$\int_0^t (t-u)^{r-1} \varphi(u)\, du = o(t^q) \qquad (t\to 0)$$

is

$$\int_0^t (t-u)^{r-1} u^k \varphi(u)\, du = o(t^{q+k}) \qquad (t\to 0),$$

where $\varphi(t)\in L$ in $0\leqq t\leqq \pi$.

This is Lemma 3 in Yano [6].

Proof of Lemma 2. For any given $\varepsilon>0$ there corresponds a number $\delta=\delta(\varepsilon)$ such as

$$\left|\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{r-1} u^k \cos \nu u \, du\right| < \varepsilon t^{q+k} \qquad (0<t\leqq \delta),$$

by assuming (2.6). And, by the assumption concerning uniform convergence we have

$$\left|\sum_{\nu=1}^{n} a_\nu \int_0^t (t-u)^{r-1} u^k \cos \nu u \, du\right| < 2\varepsilon t^{q+k}$$

for $0<\eta\leqq t\leqq \delta$ and $n\geqq n_0$, where $n_0=n_0(\eta)$. Now putting $\varphi(t)=\sum_{\nu=1}^{n} \cdot a_\nu \cos \nu u$, by the sufficiency part of Lemma 2.1 we get

$$\left|\sum_{\nu=1}^{n} a_\nu \int_0^t (t-u)^{r-1} \cos \nu u \, du\right| < C\varepsilon t^q$$

for $\eta\leqq t\leqq \delta$ and $n\geqq n_0$, where C is a constant depending on r, q and k only (cf. the proof of Lemma 2.1). In particular we have

$$\left|\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{r-1} \cos \nu u \, du\right| \leqq C\varepsilon t^q \qquad (\eta \leqq t \leqq \delta), \tag{2.7$'$}$$

which holds clearly for every $\eta>0$ by the definition of n_0. Hence we see that (2.7)$'$ holds for $0<t\leqq\delta$, and we get (2.7). This proves the lemma.

LEMMA 3. Let $-1\leqq c$, $b<c<\gamma<\beta$, $r=1+(c\beta-b\gamma)/(\beta-b+c-\gamma)$, and let the series in

$$G(t)=\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{r-1} u^k \cos \nu u \, du,$$

where k is an integer such as $k>\gamma+1$, converge uniformly in some interval $0\leqq t\leqq t_0$. In these circumstances, if

$$\sum_{\nu=1}^{n} |s_\nu^\beta| = o(n^{r+1}) \quad \text{and} \quad \sum_{\nu=n}^{2n} (|s_\nu^b| - s_\nu^b) = O(n^{c+1})$$

as $n\to\infty$, then $G(t)=o(t^{r+k})$ as $t\to 0$.

This is Corollary 4.3 in the cited paper [6].

LEMMA 4. If $r>0$ is arbitrary and $a+b\geqq[r-0]$, then

$$\int_0^t (t-u)^{r-1} u^a \left(2 \sin \frac{1}{2} u\right)^b \cos((n+A)u+B) \, du = O(t^{a+b}/n^r),$$

A and B being constants, holds uniformly in n and t such as $0<t\leqq\pi$.

This is Lemma 4 in loc. cit. [6].

3. Proof of Theorem 1. By Lemma 1, it is sufficient to show that

$$\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{p-1} \cos \nu u \, du = o(t^p) \qquad (t\to 0), \tag{3.1}$$

under the conditions in the theorem, i.e.

$$\sum_{\nu=1}^{n} |s_\nu^\beta| = o(n^{r+1}), \tag{1.1}$$

$$\sum_{\nu=n}^{2n} (|s_\nu^b|) - s_\nu^b) = O(n^{b+\delta+1}), \tag{1.2}$$

where

$$-1\leqq b, \quad b<p-1<\gamma<\beta, \quad \delta=(\beta-\gamma)(p-1-b)/(\beta-p+1). \tag{3.2}$$

Now, as Lemma 2 in Yano [5] we see that (1.1), (1.2) and (3.2) imply

$$\sum_{\nu=1}^{n} |s_\nu^b| = O(n^{b+\delta+1}). \tag{3.3}$$

Observing that $b\geqq-1$ and $\delta>0$, clearly (3.3) implies $\sum_{\nu=1}^{n} |a_\nu| = O(n^{b+\delta+1})$, and then $\sum_{\nu=n}^{\infty} |a_\nu|/\nu^p = O(n^{b+\delta+1-p})$, which is $o(1)$ as $n\to\infty$, since $b+\delta+1-p=-(p-1-b)(\gamma-p+1)/(\beta-p+1)<0$ by (3.2). In particular we have

$$\sum_{\nu=1}^{\infty} |a_\nu|/\nu^p < \infty. \tag{3.4}$$

On the other hand, letting $c=b+\delta$ and $r=p$, the conditions in (3.2) satisfy those in Lemma 3, i.e.

$$-1\leqq c, \quad b<c<\gamma<\beta, \quad r=1+(c\beta-b\gamma)/(\beta-b+c-\gamma),$$

and so by this Lemma 3, (1.1) and (1.2) imply

$$\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{p-1} u^k \cos \nu u \, du = o(t^{p+k}) \qquad (k>\gamma+1), \tag{3.5}$$

provided that the left hand side series converges uniformly in $0 \leqq t \leqq \pi$. And this assumption is satisfied since

$$\sum_{\nu=1}^{\infty} \left| a_\nu \int_0^t (t-u)^{p-1} u^k \cos \nu u \, du \right| = \sum_{\nu=1}^{\infty} |a_\nu| \cdot O(t^k/\nu^p) < \infty,$$

by Lemma 4 and (3.4). Further, (3.5) then implies (3.1) by Lemma 2 with $r=q=p$. Thus we get the theorem.

4. Riemann-Cesàro summability. A series $\sum a_\nu$ is said to be summable to s by Riemann-Cesàro method of order p and index α, or briefly summable (R, p, α) to s, if the series in

$$F(t) = (C_{p,\alpha})^{-1} t^{\alpha+1} \sum_{\nu=1}^{\infty} s_\nu^\alpha \left(\frac{\sin \nu t}{\nu t} \right)^p, \tag{4.1}$$

where

$$C_{p,\alpha} = \begin{cases} (\Gamma(\alpha+1))^{-1} \int_0^\infty u^{\alpha-p} (\sin u)^p \, du & (-1<\alpha<p-1) \\ \pi/2 & (\alpha=0,\ p=1) \\ 1 & (\alpha=-1), \end{cases}$$

converges in some interval $0<t<t_0$, and

$$\lim_{t\to 0} F(t) = s. \tag{4.2}$$

This summability method has been considered by Hirokawa [3, 4], and it coincides with summability (R, p) when $\alpha=-1$. In particular the above method is called summability (R_p) when $\alpha=0$.

Remark. The present author suspects that in the above definition the range of the index α may be extended to $-1 \leqq \alpha < p$ when p is odd, since then the number $C_{p,\alpha}$ is defined also for $p-1 \leqq \alpha < p$, the integral being in Cauchy sense, and moreover it is easily seen that

$$t^{\alpha+1} \sum_{\nu=1}^{\infty} A_{\nu-1}^\alpha \left(\frac{\sin \nu t}{\nu t} \right)^p \to C_{p,\alpha} \qquad (t\to 0), \tag{4.3}$$

similarly as in the cited paper [3].

We may suppose that $s=0$ in (4.2) with no loss of generality. We have the following theorem quite analogous to Theorem 1.

THEOREM 2. Let $-1 \leqq b$, $b<p-1<\gamma<\beta$ and $\delta = \dfrac{p-1-b}{\beta-p+1}(\beta-\gamma)$. If

$$\sum_{\nu=1}^{n} |s_\nu^\beta| = o(n^{\gamma+1}) \quad \text{and} \quad \sum_{\nu=n}^{2n} (|s_\nu^b| - s_\nu^b) = O(n^{b+\delta+1})$$

as $n\to\infty$, then the series $\sum a_\nu$ is summable (R, p, α) to zero, for $-1 \leqq \alpha < p-((-1)^p+1)/2$.

Proof. It is sufficient to show that

$$t^{\alpha+1} \sum_{\nu=1}^{\infty} s_\nu^\alpha \left(\frac{\sin \nu t}{\nu t} \right)^p \to 0 \qquad (t\to 0),$$

and its proof is, by Lemma 1, reduced to verify

$$\frac{t^{\alpha+1}}{t^p}\sum_{\nu=1}^{\infty} s_\nu^\alpha \int_0^t (t-u)^{p-1}\cos \nu u\, du \to 0 \qquad (t\to 0) \tag{4.4}$$

Further, (4.4) is true by Lemma 2 when

$$\frac{t^{\alpha+1}}{t^{p+k}}\sum_{\nu=1}^{\infty} s_\nu^\alpha \int_0^t (t-u)^{p-1}u^k\cos \nu u\, du \to 0, \tag{4.5}$$

where k is an integer such as $k>p$, provided that the series in (4.5) converges uniformly in every interval $0<\eta\leqq t\leqq\pi$. And the last assumption is satisfied by the permissibility of the succeeding transformation.

Now, using the argument in the proof of Theorem 1 of Yano [5], (4.5) may be transformed to that in

$$\begin{aligned}\frac{t^{\alpha+1}}{t^{p+k}}\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{p-1}u^k\left(2\sin\frac{1}{2}u\right)^{-(\alpha+1)}\\ \cos\left(\nu u-\frac{1}{2}(\alpha+1)(u-\pi)\right)du \to 0,\end{aligned} \tag{4.6}$$

under the assumption in the theorem, not depending on the value of α. And, (4.6) may be proved quite analogously as

$$\frac{1}{t^{p+k}}\sum_{\nu=1}^{\infty} a_\nu \int_0^t (t-u)^{p-1}u^k\cos \nu u\, du \to 0 \tag{4.7}$$

does, provided that $k-\alpha-1\geqq p$ which is permissible since k may be as large as we wish. But, as it is seen in the proof of Theorem 1, (4.7) is a result from the assumption in the theorem. Thus we get the theorem.

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