46. *Some Remarks on Inner Product in Product Space of Unitary Spaces*

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1. Let V be a unitary space over reals or complex numbers, and (x, y) be the inner product defined in it. It is known that inner product can be defined in the tensor product $V^r = V\otimes\cdots\otimes V$ (r factors in number) which satisfies: [2, 3]**)

$$(x_1\otimes x_2\otimes\cdots\otimes x_r,\ y_1\otimes y_2\otimes\cdots\otimes y_r)=(x_1, y_1)(x_2, y_2)\cdots(x_r, y_r).$$

This function, when restricted to the subspace of alternate elements $\mathcal{A}(V^r)$ and the subspace of symmetric elements $\mathcal{S}(V^r)$ of V^r, gives rise respectively to inner product of the space of exterior r-vectors $\Lambda^r(V)$ and $P^r(V)$ (to be defined below), since these spaces are respectively isomorphic to $\mathcal{A}(V^r)$ and $\mathcal{S}(V^r)$.

If u is the conjugate isomorphism between V and its dual (conjugate) space V^*, then

$$\langle x, u(y)\rangle=(x, y) \qquad \text{for all } x\in V,$$

where $\langle x, y^*\rangle$ is the pairing of V and V^* to scalars.

Denote by $u^r: V^r\to V_r = V^*\otimes\cdots\otimes V^*$ (r factors in number) the r-th tensor power of u, then u^r is an isomorphism between V^r and V_r and

$$u^r(x_1\otimes\cdots\otimes x_r)=u(x_1)\otimes\cdots\otimes u(x_r).$$

Moreover, if $\Lambda^r u: \Lambda^r(V)\to\Lambda^r(V^*)$ is the r-th exterior power of u, then $\Lambda^r u$ is an isomorphism between $\Lambda^r(V)$ and $\Lambda^r(V^*)$ and

$$(\Lambda^r u)(x_1\wedge\cdots\wedge x_r)=u(x_1)\wedge\cdots\wedge u(x_r).$$

As it is known that $(V^r)^*\approx V_r$ and $(\Lambda^r(V))^*\approx\Lambda^r(V^*)$, we can identify the isomorphic spaces.

Now, we propose to show:

Theorem 1. *u^r is the conjugate isomorphism between V^r and $V_r=(V^r)^*$, and $\Lambda^r u$ is the conjugate isomorphism between $\Lambda^r(V)$ and $\Lambda^r(V^*)=(\Lambda^r(V))^*$.*

Proof. For any $x_1\otimes\cdots\otimes x_r$ and $y_1\otimes\cdots\otimes y_r$ in V^r, we have

$$\begin{aligned}&\langle x_1\otimes\cdots\otimes x_r\ \ u^r(y_1\otimes\cdots\otimes y_r)\rangle\\ &=\langle x_1\otimes\cdots\otimes x_r\ \ u(y_1)\otimes\cdots\otimes u(y_r)\rangle\\ &=\langle x_1\ u(y_1)\rangle\cdots\langle x_r\ u(y_r)\rangle\end{aligned}$$

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**) In the sequel we follow the notation of S. S. Chern [2]. The number in bracket denotes the references at the end of this paper.

$$=(x_1\ y_1)\cdots(x_r\ y_r)$$
$$=(x_1\otimes\cdots\otimes x_r\ \ y_1\otimes\cdots\otimes y_r).$$

The general relation $\langle x\ u(y)\rangle=(x\ y)$ follows from the bilinearity of $\langle x\ y\rangle$ and linearity of u^r.

Next, under the isomorphism between $\Lambda^r(V)$ and $\mathcal{A}(V^r)$, $x_1\wedge\cdots\wedge x_r$ corresponds to $A_r(x_1\otimes\cdots\otimes x_r)$ where A_r is the alternation in V^r. The inner product in V^r gives rise to the following definition of inner product in $\Lambda^r(V)$:

$$\begin{aligned}
&(x_1\wedge\cdots\wedge x_r\ \ y_1\wedge\cdots\wedge y_r)\\
&=(1/r!)(A_r(x_1\otimes\cdots\otimes x_r)\ \ A_r(y_1\otimes\cdots\otimes y_r))\\
&=(1/r!)\,(\sum_\sigma(\operatorname{sgn}\sigma)x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}\ \ \sum_\tau(\operatorname{sgn}\tau)y_{\tau(1)}\otimes\cdots\otimes y_{\tau(r)})\\
&=(1/r!)\sum_\sigma(\operatorname{sgn}\sigma)\sum_\tau(\operatorname{sgn}\tau)(x_{\sigma(1)}\ y_{\tau(1)})\cdots(x_{\sigma(r)}\ y_{\tau(r)})\\
&=(1/r!)\sum_\sigma(\operatorname{sgn}\sigma)\begin{vmatrix}(x_{\sigma(1)}\ y_1)\cdots(x_{\sigma(1)}\ y_r)\\ \cdots\cdots\\ (x_{\sigma(r)}\ y_1)\cdots(x_{\sigma(r)}\ y_r)\end{vmatrix}\\
&=\begin{vmatrix}(x_1\ y_1)\cdots(x_1\ y_r)\\ \cdots\cdots\\ (x_r\ y_1)\cdots(x_r\ y_r)\end{vmatrix}=|(x_i\ y_k)|,
\end{aligned}$$

where $x_1\wedge\cdots\wedge x_r$, $y_1\wedge\cdots\wedge y_r\ \varepsilon\Lambda^r(V)$.

On the other hand,

$$\begin{aligned}
&\langle x_1\wedge\cdots\wedge x_r\ \ (\Lambda^r u)(y_1\wedge\cdots\wedge y_r)\rangle\\
&=\langle x_1\wedge\cdots\wedge x_r\ \ u(y_1)\wedge\cdots\wedge u(y_r)\rangle\\
&=|\langle x_i\ u(y_k)\rangle|=|(x_i\ y_k)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle x_1\wedge\cdots\wedge x_r\ \ (\Lambda^r u)(y_1\wedge\cdots\wedge y_r)\rangle\\
&=(x_1\wedge\cdots\wedge x_r\ \ y_1\wedge\cdots\wedge y_r).
\end{aligned}$$

The general relation follows from the bilinearity of $\langle x\ y\rangle$ and the linearity of $\Lambda^r u$.

2. It is of interest to take care of the relation between the above things and classical treatment of tensor analysis.

Let $(e_1, e_2,\cdots, e_n)$ and $(e'^1, e'^2,\cdots, e'^n)$ be dual bases in V and V^*. It is obvious that $e_{i_1}\otimes\cdots\otimes e_{i_r}$ and $e_{i_1}\wedge\cdots\wedge e_{i_r}$ $(i_1<i_2<\cdots<i_r)$ are respectively the basis in V^r and $\Lambda^r(V)$. If $y=\eta^j e_j$, $u(y)=\eta_i e'^i$ (summation convention is used), then $\eta_i=g_{ij}\eta^j$, where $g_{ij}=(e_i\ e_j)$. Consequently, if $y=t^{j_1j_2\cdots j_r}e_{j_1}\otimes\cdots\otimes e_{j_r}\ \varepsilon V^r$ and $u(y)=t_{i_1i_2\cdots i_r}e'^{i_1}\otimes\cdots\otimes e'^{i_r}\ \varepsilon V_r$, then

$$t_{i_1i_2\cdots i_r}=g_{i_1j_1}g_{i_2j_2}\cdots g_{i_rj_r}t^{j_1j_2\cdots j_r}.$$

Moreover, if $y=\sum_{j_1<\cdots<j_r}t^{(j_1j_2\cdots j_r)}e_{j_1}\wedge\cdots\wedge e_{j_r}\ \varepsilon\Lambda^r(V)$ and $u(y)=\sum_{i_1<\cdots<i_r}t_{(i_1i_2\cdots i_r)}e'^{i_1}\wedge\cdots\wedge e'^{i_r}\ \varepsilon\Lambda^r(V^*)$, then

$$t_{(i_1i_2\cdots i_r)}=\sum_{j_1<j_2<\cdots<j_r}\begin{vmatrix}g_{i_1j_1}\cdots g_{i_1j_r}\\ \cdots\cdots\\ g_{i_1j_1}\cdots g_{i_rj_r}\end{vmatrix}t^{(j_1j_2\cdots j_r)},$$

and these are respectively the covariant and contravariant components of tensors obtained by identifying the corresponding elements under the conjugate isomorphism $V^r \approx V_r$ or $\Lambda^r(V) \approx \Lambda^r(V^*)$.

Assume that V is a euclidean vector space, then $(e_1\wedge\cdots\wedge e_n\ e_1\wedge\cdots\wedge e_n)=|g_{ij}|=g$, where $g_{ij}=(e_i\ e_j)$. The unit elements $e=(1/\sqrt{g})e_1\wedge\cdots\wedge e_n$ and $e'=\sqrt{g}\,e'^1\wedge\cdots\wedge e'^n$ form respectively the basis of one dimensional vector spaces $\Lambda^n(V)$ and $\Lambda^n(V^*)$. Under the isomorphisms $\Lambda^n(V)\to\mathcal{A}(V^n)$ and $\Lambda^n(V^*)\to\mathcal{A}(V_n)$, e and e' respectively corresponds

$$\eta^{i_1i_2\cdots i_n}e_{i_1}\otimes\cdots\otimes e_{i_n},\quad \eta^{i_1i_2\cdots i_n}=(1/\sqrt{g})\epsilon^{i_1i_2\cdots i_n},$$

and

$$\eta_{i_1i_2\cdots i_n}e'^{i_1}\otimes\cdots\otimes e'^{i_n},\quad \eta_{i_1i_2\cdots i_n}=\sqrt{g}\,\epsilon_{i_1i_2\cdots i_n},$$

where $\epsilon^{i_1i_2\cdots i_n}=\epsilon_{i_1i_2\cdots i_n}=\begin{cases}1 & \text{if } (i_1,\cdots,i_n) \text{ is an even permutation,}\\ -1 & \text{if } (i_1,\cdots,i_n) \text{ is an odd permutation.}\end{cases}$

It is known that the linear map $\varphi:\Lambda^r(V)\to\Lambda^{n-r}(V^*)$ defined by

$$\varphi(x)=x\,\lrcorner\, e',\qquad x\in\Lambda^r(V)$$

and

$$\langle z\ x\,\lrcorner\, e'\rangle=\langle z\wedge x\ e'\rangle,\qquad z\in\Lambda^{n-r}(V)$$

is an isomorphism onto.

We should like to note that if

$$x=\sum_{i_1<i_2<\cdots<i_r} t^{i_1i_2\cdots i_r}e_{i_1}\wedge\cdots\wedge e_{i_r},$$

then

$$x\,\lrcorner\, e'=\sum_{j_1<j_2<\cdots<j_{n-r}} t_{j_1j_2\cdots j_{n-r}}e'^{j_1}\wedge\cdots\wedge e'^{j_{n-r}},$$

where

$$t_{j_1j_2\cdots j_{n-r}}=(1/r!)\sum_{(i)} t^{i_1i_2\cdots i_r}\eta_{j_1j_2\cdots j_{n-r}i_1i_2\cdots i_r}.$$

So $x\,\lrcorner\, e'$ corresponds to an alternate tensor $t_{j_1j_2\cdots j_{n-r}}$ which is essentially the adjoint tensor [4] of the alternate tensor $t^{i_1i_2\cdots i_r}$ corresponding to x.

It is also obvious that if $(e_1,e_2,\cdots,e_n)$ is a set of orthonormal basis in V, then $e_{i_1}\otimes\cdots\otimes e_{i_r}$ and $e_{i_1}\wedge\cdots\wedge e_{i_r}$ $(i_1<i_2<\cdots<i_r)$ are respectively the orthonormal basis of V^r and $\Lambda^r(V)$.

A decomposable element (or multilinear vector) $x_1\wedge\cdots\wedge x_r$ in $\Lambda^r(V)$ determines an r-simplex $(P_0,P_1,\cdots,P_r)$ when $x_i=\overrightarrow{P_0P_i}$ in euclidean n-space. Then the volume of this r-simplex is $1/r!$ of the length of $x_1\wedge\cdots\wedge x_r$ in the sense of metric induced by the inner product in $\Lambda^r(V)$ mentioned above. Because

$$(x_1\wedge\cdots\wedge x_r\ x_1\wedge\cdots\wedge x_r)=|(x_i\ x_k)|$$
$$=\sum t^{(i_1i_2\cdots i_r)}\overline{t^{(i_1i_2\cdots i_r)}}=\sum\{t^{(i_1i_2\cdots i_r)}\}^2,$$

provided $x_1\wedge\cdots\wedge x_r=t^{(i_1i_2\cdots i_r)}e_{i_1}\wedge\cdots\wedge e_{i_r}$ is referred to the orthonormal basis.

3. Before discussing on conjugate isomorphism in the case of $P^r(V)$, we do some preparation on the properties of $P^r(V)$. This can be done completely parallel to the case of $\Lambda^r(V)$ [1, 2].

Denote by M^r the kernel of symmetric linear map $V^r\to V^r$ and put $P^r(V)=V^r/M^r$. If $\varphi:V^r\to P^r(V)$ is the natural projection, sending an element of V^r to its coset mod M^r, we shall use the notation:

$$\varphi(x_1 \otimes \cdots \otimes x_r) = x_1 \cdots x_r.$$

It can be easily shown that a linear map $f: V^r \to Z$, where Z is a vector space, is symmetric if and only if $f(M^r)=0$. As a corollary of this theorem, it follows that the space $\mathcal{S}(V^r; Z)$ of all symmetric linear maps $f: V^r \to Z$ is isomorphic with the space $\mathcal{L}(P^r(V); Z)$ of all linear maps $g: P^r(V) \to Z$, the correspondence being established by the relation $f = g\varphi$. Therefore, if $k: \mathcal{L}(P^r(V); F) \to \mathcal{S}(V^r; F)$ is the isomorphism, then for $\theta = k^{-1}(\theta)\varphi \in \mathcal{S}(V^r; F)$ we have:

$$\theta(x_1 \otimes \cdots \otimes x_r) = \langle x_1 \cdots x_r \quad k^{-1}(\theta)\rangle.$$

Let M_r be the kernel of symmetric map $S_r: V_r \to V_r$, and $P^r(V^*) = V_r/M_r$, then the range of S_r is the space of symmetric covariant r-tensors $\mathcal{S}(V_r)$, and S_r induces an isomorphism $\bar{k}$ of the space $P^r(V^*)$ onto $\mathcal{S}(V_r)$ such that $\bar{k}\varphi = S_r$. Therefore

$$S_r(x'_1 \otimes \cdots \otimes x'_r) = \bar{k}(x'_1 \cdots x'_r).$$

For any $\theta \in \mathcal{S}(V^r; F)$, there exists an element $z'(\theta) \in V_r[\approx (V^r)^*]$ such that

$$\langle x_1 \otimes \cdots \otimes x_r \quad z'(\theta)\rangle = \theta(x_1 \otimes \cdots \otimes x_r)$$

for $x_1 \otimes \cdots \otimes x_r \in V^r$. It is easily shown that $z'(\theta)$ is symmetric element in V_r and that the map $z': \theta \to z'(\theta)$ is an isomorphism from $\mathcal{S}(V^r; F)$ onto $\mathcal{S}(V_r)$.

From the above discussion we have the following diagram:

$$(P^r(V))^* = \mathcal{L}(P^r(V); F) \underset{\overset{k}{\rightarrow}}{\approx} (V^r; F) \underset{\overset{z'}{\rightarrow}}{\approx} (V_r) \underset{\overset{\bar{k}}{\leftarrow}}{\approx} P^r(V^*).$$

Thus we have

$$P^r(V^*) \approx (P^r(V))^*.$$

Let $i = k^{-1}z'^{-1}\bar{k}$ be the composed isomorphism: $P^r(V^*) \to (P^r(V))^*$. If we put $\theta = z'^{-1}\bar{k}(x'_1 \cdots x'_r)$ where $x'_1 \cdots x'_r \in P^r(V^*)$, we have $z'(\theta) = \bar{k}(x'_1 \cdots x'_r) = S_r(x'_1 \otimes \cdots \otimes x'_r)$. Consequently

$$\begin{aligned}
&\langle x_1 \cdots x_r \quad k^{-1}z'^{-1}\bar{k}(x'_1 \cdots x'_r)\rangle \\
&= \langle x_1 \cdots x_r \quad k^{-1}(\theta)\rangle \\
&= \theta(x_1 \otimes \cdots \otimes x_r) \\
&= \langle x_1 \otimes \cdots \otimes x_r \quad z'(\theta)\rangle \\
&= \langle x_1 \otimes \cdots \otimes x_r \quad S_r(x'_1 \otimes \cdots \otimes x'_r)\rangle \\
&= \langle x_1 \otimes \cdots \otimes x_r \quad \sum_\sigma x'_{\sigma(1)} \otimes \cdots \otimes x'_{\sigma(r)}\rangle \\
&= \sum_\sigma \langle x_1 \quad x'_{\sigma(1)}\rangle \langle x_2 \quad x'_{\sigma(2)}\rangle \cdots \langle x_r \quad x'_{\sigma(r)}\rangle.
\end{aligned}$$

Thus, if we identify the corresponding elements under the isomorphism $i = k^{-1}z'^{-1}\bar{k}$, then we have

$$\begin{aligned}
&\langle x_1 \cdot x_2 \cdots x_r \quad x'_1 \cdot x'_2 \cdots x'_r\rangle \\
&= \sum_\sigma \langle x_1 \quad x'_{\sigma(1)}\rangle \langle x_2 \quad x'_{\sigma(2)}\rangle \cdots \langle x_r \quad x'_{\sigma(r)}\rangle.
\end{aligned}$$

Next, let $u: V \to V^*$ be the conjugate isomorphism, and $u^r: V^r \to V_r$

be the r-th power of u. If $\eta: V_r \to P^r(V^*)$ be the natural projection, then $\eta u^r: V^r \to P^r(V^*)$ is a symmetric linear map. Then there exists a linear map $P^r u: P^r(V) \to P^r(V^*)$ such that $(P^r u)\circ\varphi = \eta\circ u^r$, and

$$(P^r u)(x_1\cdots x_r) = u(x_1)\cdots u(x_r).$$

Now we have the following:

Theorem 2. $P^r u$ is the conjugate isomorphism between $P^r(V)$ and $P^r(V^*)$ [identified with $(P^r(V))^*$].

Proof. Under the isomorphism between $P^r(V)$ and $\mathcal{S}(V^r)$, $x_1\cdots x_r$ corresponds to $S_r(x_1\otimes\cdots\otimes x_r)$. And the inner product in V^r gives rise to the following definition of inner product in $P^r(V)$.

$$\begin{aligned}
&(x_1\cdots x_r \quad y_1\cdots y_r)\\
&=(1/r!)(S_r(x_1\otimes\cdots\otimes x_r) \quad S_r(y_1\otimes\cdots\otimes y_r))\\
&=(1/r!)\,(\sum_\sigma x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}\ \sum_\tau y_{\tau(1)}\otimes\cdots\otimes y_{\tau(r)})\\
&=(1/r!)\sum_\sigma\sum_\tau (x_{\sigma(1)} \ y_{\tau(1)})\cdots(x_{\sigma(r)} \ y_{\tau(r)})\\
&=\sum_\tau (x_1 \ y_{\tau(1)})\cdots(x_r \ y_{\tau(r)}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\langle x_1\cdots x_r \quad (P^r u)(y_1\cdots y_r)\rangle\\
&=\langle x_1\cdots x_r \quad u(y_1)\cdots u(y_r)\rangle\\
&=\sum_\tau \langle x_1 \ u(y_{\tau(1)})\rangle\cdots\langle x_r \ u(y_{\tau(r)})\rangle\\
&=\sum_\tau (x_1 \ y_{\tau(1)})\cdots(x_r \ y_{\tau(r)}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(x_1\cdots x_r \quad y_1\cdots y_r)\\
&=\langle x_1\cdots x_r \quad (P^r u)(y_1\cdots y_r)\rangle.
\end{aligned}$$

The general relation follows from the bilinearity of $\langle x \ y\rangle$ and the linearity of $P^r u$.

From the above discussion it is also easily seen that the following relations can be respectively used as the definition of inner product in V^r, $\Lambda^r(V)$ and $P^r(V)$:

$$\begin{aligned}
(x \ y) &= \langle x \ u^r(y)\rangle \ ; \quad x, y \in V^r,\\
(x \ y) &= \langle x \ (\Lambda^r u)y\rangle; \quad x, y \in \Lambda^r(V),\\
(x \ y) &= \langle x \ (P^r u)y\rangle; \quad x, y \in P^r(V).
\end{aligned}$$

4. It is well known that the Grassmann algebra

$$\Lambda(V) = \Lambda^0(V) + \Lambda^1(V) + \cdots + \Lambda^n(V) \qquad \text{(direct sum)}$$

is a vector space of dimension 2^n and that $\Lambda(V)^* \approx \Lambda(V^*)$, where V^* is the dual space of V. Moreover, if we identify the corresponding elements under this isomorphism, the pairing of these two spaces satisfies the following [1]:

$$\langle x \ x'\rangle = \left\langle \sum_{p=0}^n x_p \ \sum_{p=0}^n x'_p \right\rangle = \sum_{p=0}^n \langle x_p \ x'_p\rangle,$$

where $x=\sum_{p=0}^n x_p \in \Lambda(V)$, $x'=\sum_{p=0}^n x'_p \in \Lambda(V^*)$ with $x_p \in \Lambda^p(V)$ and $x'_p \in \Lambda^p(V^*)$.

Let u be the conjugate isomorphism from V onto V^*. Denote $\overline{u}$ the canonic prolongment [1] of u in $\Lambda(V)$, then for $y=\sum_{p=0}^{n} y_p \varepsilon \Lambda(V)$, we have

$$\overline{u}(y)=\sum_{p=0}^{n}(\Lambda^p u)y_p,$$

where $\Lambda^p u$ is the p-th exterior power of u.

Now we define (x, y) in $\Lambda(V)$ by the following:

$$(x, y)=\langle x, \overline{u}(y)\rangle.$$

Then, we can prove easily that (x, y) is an inner product as follows:

By definition, we have

$$(x, y)=\Big(\sum_{p=0}^{n} x_p \ \sum_{p=0}^{n} y_p\Big)=\Big\langle \sum_{p=0}^{n} x_p \ \sum_{p=0}^{n} (\Lambda^p u)y_p\Big\rangle$$

$$=\sum_{p=0}^{n}\langle x_p \ (\Lambda^p u)y_p\rangle=\sum_{p=0}^{n}(x_p \ y_p),$$

where $(x_p \ y_p)$ is the inner product defined above (§ 1).

Consequently, we have

1) $\overline{(x \ y)}=\sum_{p=0}^{n}\overline{(x_p \ y_p)}=\sum_{p=0}^{n}(y_p \ x_p)=(y \ x)$.

2) Evidently $(\alpha x+\beta y, z)=\alpha(x \ z)+\beta(y \ z)$; $\alpha, \beta \varepsilon F$, $x, y, z \varepsilon \Lambda(V)$.

3) $(x \ x)=\sum_{p=0}^{n}(x_p \ x_p)$ is real and is $\geqq 0$. Moreover, as $(x_p \ x_p)=0$ if and only if $x_p=0$, so $(x \ x)=0$ if and only if $x=0$.

Thus we have the following:

Theorem 3. *With inner product $(x \ y)$ defined above, $\Lambda(V)$ is a unitary space, and the conjugate isomorphism between $\Lambda(V)$ and $\Lambda(V^*)$ is given by the canonic prolongment $\overline{u}$ of u.*

References

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