60. *On the Sets of Regular Measures. I*

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1. Introduction. Let (X, S) be a topological measurable space, and let us consider more than one measure on S. The measures of our objects are not necessarily finite. The relations among measurable sets regular with respect to a fixed measure m are well known. Here, the term "regular" is employed as usual: a measurable set E is inner regular with respect to m if

$$m(E)=\sup\{m(C): E\supseteq C, C\in \boldsymbol{C}\},$$

where $\boldsymbol{C}$ is the class of compact measurable sets. The measurable set E is outer regular with respect to m if

$$m(E)=\inf\{m(U): E\subseteq U, U\in \boldsymbol{U}\},$$

where $\boldsymbol{U}$ is the class of open measurable sets. If each measurable set is inner (outer) regular, the measure m will be inner (outer) regular.

About the relations among the regularities of two or more measures, G. Swift [2] investigated chiefly concerning irregular Borel measures, and R. E. Zink [3] concerning integral measures.

We shall now propose and make a study of the following problems:

(1) Let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of measures and ν be a measure such that $\lim_{i\to\infty}\mu_i(E)=\nu(E)(E\in S)$. Then will the inner (outer) regularities of $\mu_i(i=1, 2, \cdots)$ be preserved on ν?

(2) Let $\mu_1\smile\mu_2(\mu_1\frown\mu_2)$ be the superior (inferior) measure of the two measures μ_1 and μ_2. Then will the inner (outer) regularities of μ_1 and μ_2 be preserved on $\mu_1\smile\mu_2(\mu_1\frown\mu_2)$? Next, does the argument change when we substitute a set of measures $\{\mu_\lambda\}_{\lambda\in\Lambda}$ (of arbitrary numbers) in place of μ_1, μ_2?

(3) Let f_1 and f_2 be two non-negative measurable functions and μ be a measure. Let us define the measures μ_1, μ_2 and ν by means of the equations

$$\mu_1=\int f_1 d\mu, \quad \mu_2=\int f_2 d\mu, \quad \nu=\int \sqrt{f_1 f_2}\, d\mu.$$

Under what conditions will the inner (outer) regularities of μ_1 and μ_2 be induced on ν?

(4) The similar problems with respect to irregular measures

If we deal with finite measures only, the arguments will be very simple, but on the contrary the permission of introducing infinite measures complicates the affairs, because, for instance, $m(E)=\infty$

necessarily implies the outer regularity of E with respect to m (not necessarily the strictly outer regularity of E), having no effect to the demonstration consequently.

The definitions of the above-mentioned "strictly outer regular" (and "strictly inner regular") are as follows [3]: a measurable set E will be termed strictly outer regular with respect to m if and only if

$$\inf\{m(U-E) : E \subseteq U,\ U \in \boldsymbol{U}\}=0,$$

and strictly inner regular with respect to m if and only if

$$\inf\{m(E-C) : E \supseteq C,\ C \in \boldsymbol{C}\}=0,$$

respectively.

2. Sequence of measures. Throughout this section, let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of measures and ν be a measure such that $\lim_{i\to\infty} \mu_i(E)=\nu(E)(E \in \boldsymbol{S})$.

Theorem 1. If a set $E \in \boldsymbol{S}$ is inner regular with respect to $\mu_i(i=1,2,\cdots)$, then E is inner regular with respect to ν also.

Proof. If $\nu(E)<\infty$, there exist a positive integer i_0 and a sequence, $\{C_i\}_{i=i_0}^{\infty}$, of compact measurable sets such that $\mu_i(E)<\infty$, $C_i \subseteq E$ and $\mu_i(C_i) \geqq \mu_i(E)-\frac{1}{i}(i \geqq i_0)$. Let $C=\bigcup_{i=i_0}^{\infty} C_i$, then $C \subseteq E$ and $\mu_i(C) \geqq \mu_i(E)-\frac{1}{i}(i \geqq i_0)$. Therefore, $\nu(C)=\lim_{i\to\infty} \mu_i(C) \geqq \lim_{i\to\infty} \mu_i(E)=\nu(E)$, hence $\nu(C)=\nu(E)$. Let $\varepsilon>0$ be specified. There, then, exists an integer N_0 for which $\nu(\bigcup_{i=i_0}^{N_0} C_i) \geqq \nu(E)-\varepsilon$, and accordingly E is inner regular with respect to ν.

On the other hand, if $\nu(E)=\infty$, for an arbitrary $M>0$, there exist an integer i_1 and a sequence, $\{C_i\}_{i=i_1}^{\infty}$, of compact measurable sets such that $\mu_i(E) \geqq 3M$, $C_i \subseteq E$ and $\mu_i(C_i) \geqq 2M(i \geqq i_1)$. Let $C=\bigcup_{i=i_1}^{\infty} C_i$, then $C \subseteq E$ and $\mu_i(C) \geqq 2M(i \geqq i_1)$. Therefore $\nu(C) \geqq 2M$, and for a suitable integer N_1, $\nu(\bigcup_{i=i_1}^{N_1} C_i) \geqq M$. Thus the inner regularity of E with respect to ν results.

In the case of outer regularities, the situations are not parallel to the above.

Theorem 2. If a set $E \in \boldsymbol{S}$ is outer regular with respect to $\mu_i(i=1,2,\cdots)$ and $\nu(E)<\infty$, then E is outer regular with respect to ν if and only if there exists a measurable open set U containing E such as $\nu(U)<\infty$. (That ν is a finite measure surely satisfies the above condition following "only if".)

Proof. Necessity. Clear

Sufficiency. There exists a sequence, $\{U_i\}_{i=1}^{\infty}$, of open measurable sets such that $U_i \supseteq E$ and $\mu_i(U_i) \leqq \mu_i(E)+\frac{1}{i}(i \geqq 1)$. Let $V_i=U_i U$, and then $V_i \supseteq E$, $\mu_i(V_i) \leqq \mu_i(E)+\frac{1}{i}(i \geqq 1)$. Let us consider now the measurable set $V=\bigcap_{i=1}^{\infty} V_i$. We have $V \supseteq E$, $\mu_i(V) \leqq \mu_i(E)+\frac{1}{i}(i \geqq 1)$, hence $\nu(V) \leqq \nu(E)$, $\nu(V)=\nu(E)$. Now by the assumptions, it follows that

$\nu(V_1)<\infty$ and $\lim_{n\to\infty}\nu(\bigcap_{i=1}^{n} V_i)=\nu(V)$, therefore there exists an integer N for an arbitrary $\varepsilon>0$ such that $\nu(\bigcap_{i=1}^{N} V_i)\leqq\nu(V)+\varepsilon=\nu(E)+\varepsilon$, here $\bigcap_{i=1}^{N} V_i$ being an open measurable set containing E.

Corollary 1. If a set $E\in\boldsymbol{S}$ is outer regular with respect to $\mu_i(i=1, 2, \cdots)$ and $\boldsymbol{D}$ is the class of all open measurable sets U of ν-infinite measure, containing E, then a sufficient condition of the outer regularity of E with respect to ν is that the sequence of numbers, $\left\{\frac{1}{\mu_i(U)}\right\}_{i=1}^{\infty}$, converges to zero uniformly regarding the class $\boldsymbol{D}$. (When the class $\boldsymbol{D}$ consists of sets of finite numbers, the above condition is sufficiently satisfied.)

Proof. We need consider only the case of ν-finite E. Then, there exist an integer i_0 such that $\mu_i(E)\leqq\nu(E)+1<\infty(i\geqq i_0)$, and accordingly a sequence, $\{U_i\}_{i=i_0}^{\infty}$, of open measurable sets such that $U_i\supseteq E$ and $\mu_i(U_i)\leqq\nu(E)+2<\infty(i\geqq i_0)$. Now we shall show the existence of at least one integer i for which $\nu(U_i)<\infty$. If contrarily $\nu(U_i)=\infty$ for all $i\geqq i_0$, there would exist an integer j_0 independent of i such that $\mu_j(U_i)>\nu(E)+2(i\geqq i_0, j\geqq j_0)$. Let $k_0=\text{Max}\,(i_0, j_0)$. The above inequality would imply $\mu_{k_0}(U_{k_0})>\nu(E)+2$, contradiction to the definition of $\{U_i\}_{i=i_0}^{\infty}$.

Corollary 2. If a set $E\in\boldsymbol{S}$ is outer regular with respect to $\mu_i(i=1, 2, \cdots)$ and $\mu_i(U)\geqq\nu(U)(U\supseteq E, U\in\boldsymbol{U}, i=1, 2, \cdots)$, then E is outer regular with respect to ν. (When the sequence $\{\mu_i(U)\}_{i=1}^{\infty}$ decreases for every open measurable set U, the above second condition is naturally satisfied.)

Proof. For a set $U\in\boldsymbol{D}$, it follows that $\mu_i(U)=\infty$ and $\frac{1}{\mu_i(U)}=0$ $(i=1, 2, \cdots)$ by the assumption. Thus the argument will be reduced to Corollary 1.

3. Superior and inferior measures. Let μ_1 and μ_2 be arbitrary two measures. Let us define the non-negative set function ν on $\boldsymbol{S}$ as follows: $\nu(E)=\sup\{\mu_1(A_1)+\mu_2(A_2): A_1\smile A_2=E, A_1\frown A_2=\theta, A_1\in\boldsymbol{S}, A_2\in\boldsymbol{S}\}$ $(E\in\boldsymbol{S})$. Then, it is easily verified that ν is a measure, in respect of which $\mu_1\leqq\nu$, $\mu_2\leqq\nu$ hold, and $\mu_1\leqq\mu$, $\mu_2\leqq\mu$ imply $\nu\leqq\mu$ for any measure μ. This measure ν is called the superior measure of the two measures, μ_1 and μ_2. The entirely similar methods give the definitions of the superior measures of a sequence of measures, $\{\mu_i\}_{i=1}^{\infty}$, and a set of measures, $\{\mu_\lambda\}_{\lambda\in\Lambda}$ (of arbitrary numbers), respectively, that is, the superior measure ν of $\{\mu_i\}_{i=1}^{\infty}$ is defined such as $\nu(E)=\sup\{\sum_{i=1}^{\infty}\mu_i(A_i): \bigcup_{i=1}^{\infty}A_i=E, A_j\frown A_k=\theta\ (j\neq k), A_i\in\boldsymbol{S}\,(i=1, 2, \cdots)\}\,(E\in\boldsymbol{S})$ and that of $\{\mu_\lambda\}_{\lambda\in\Lambda}$, such as $\nu(E)=\sup\{\sum_{i=1}^{\infty}\mu_{\lambda_i}(A_i): \bigcup_{i=1}^{\infty}A_i=E, A_j\frown A_k=\theta\ (j\neq k), A_i\in\boldsymbol{S}\ (i=1, 2,\cdots), \lambda_i\in\Lambda(i=1, 2, \cdots)\}$ $(E\in\boldsymbol{S})$.

In the next, the substitution of "inf." for "sup." in the above will define the inferior measures of the two measures, μ_1, μ_2, a sequence of measures $\{\mu_i\}_{i=1}^{\infty}$ and a set of measures $\{\mu_\lambda\}_{\lambda\in\Lambda}$, respectively.

In particular, if the two measures, μ_1 and μ_2, are formed as follows, $\mu_1(E)=\int_E f_1 d\mu$, $\mu_2(E)=\int_E f_2 d\mu (E\in S)$, by means of two non-negative measurable functions f_1, f_2 and a measure μ, then the superior and inferior measures of μ_1 and μ_2 will evidently be equal to $\int (f_1 \smile f_2)d\mu$ and $\int (f_1 \frown f_2)d\mu$, respectively.

Henceforth, the symbols $\smile$ and $\frown$ will be used to denote the superior and inferior measures, respectively.

Theorem 3. (1) Let $\nu=\mu_1\smile\mu_2$ be the superior measure of μ_1 and μ_2. Then, if a certain set $E\in S$ is inner (outer) regular with respect to μ_1 and μ_2, E is inner (outer) regular with respect to ν, too.

(2) Let $\nu=\bigcup_{i=1}^{\infty}\mu_i$ be the superior measure of $\{\mu_i\}_{i=1}^{\infty}$. Then, if a set $E\in S$ is inner regular with respect to μ_i $(i=1,2,\cdots)$, E is inner regular with respect to ν, too.

(3) Let $\nu=\bigcup_{\lambda\in\Lambda}\mu_\lambda$ be the superior measure of $\{\mu_\lambda\}_{\lambda\in\Lambda}$. Then, if every μ_λ, $\lambda\in\Lambda$ is inner regular, ν is also inner regular.

Proof. (1) Let a set $E\in S$ be inner regular with respect to μ_1 and μ_2.

Suppose first that $\nu(E)<\infty$. By virtue of the inequalities $\mu_1\leqq\nu$, $\mu_2\leqq\nu$, it holds that $\mu_1(E)<\infty$, $\mu_2(E)<\infty$, therefore there exist the two compact measurable sets C_1, C_2 such that $C_1\subseteq E$, $C_2\subseteq E$ and $\mu_1(E-C_1)<\varepsilon/2$, $\mu_2(E-C_2)<\varepsilon/2$ for an arbitrary $\varepsilon>0$. Thus, by the inequality $\nu\leqq\mu_1+\mu_2$, certainly $\nu(E-(C_1\smile C_2))\leqq\mu_1(E-(C_1\smile C_2))+\mu_2(E-(C_1\smile C_2))\leqq\mu_1(E-C_1)+\mu_2(E-C_2)<\varepsilon$, here $C_1\smile C_2$ being a compact set contained in E, accordingly the inner regularity of E with respect to ν results.

Now consider the case $\nu(E)=\infty$. Again, by the inequality $\nu\leqq\mu_1+\mu_2$, $\mu_1(E)=\infty$ or $\mu_2(E)=\infty$ holds. Let $\mu_1(E)=\infty$ for the present. There exists a compact set C such that $C\subseteq E$ and $\mu_1(C)>M$ for an arbitrary $M>0$, hence $\nu(C)>M$ and the inner regularity of E with respect to ν is established.

Regarding the case of the outer regularities, it will be argued almost similarly.

(2) Let a set $E\in S$ be inner regular with respect to μ_i $(i=1,2,\cdots)$. In this case also, we shall rely on the inequalities $\mu_i\leqq\nu$ $(i=1,2,\cdots)$, $\nu\leqq\mu_1+\mu_2+\cdots+\mu_i+\cdots$, and distinguish the two cases:

I. $\nu(E)<\infty$. There exists a sequence of compact measurable sets, $\{C_i\}_{i=1}^{\infty}$, such that $C_i\subseteq E$ and $\mu_i(E-C_i)<\varepsilon/2^{i+1}$ $(i=1,2,\cdots)$ for an arbitrary $\varepsilon>0$. Therefore $\nu(E-\bigcup_{i=1}^{\infty}C_i)\leqq\sum_{i=1}^{\infty}\mu_i(E-C_i)<\varepsilon/2$ and

$\nu(E-\bigcup_{i=1}^{N}C_i)\leqq\nu(E-\bigcup_{i=1}^{\infty}C_i)+\nu(\bigcup_{i=1}^{\infty}C_i-\bigcup_{i=1}^{N}C_i)<\varepsilon/2+\varepsilon/2=\varepsilon$ for a suitable integer N.

II. $\nu(E)=\infty$. If $\mu_{i_0}(E)=\infty$ holds for at least one integer i_0, there exists a compact measurable set $C\subseteqq E$ such that $\mu_{i_0}(C)>M$ for an arbitrary $M>0$, hence $\nu(C)>M$.

On the other hand, if $\mu_i(E)<\infty$ $(i=1, 2, \cdots)$, there exists a sequence of compact measurable sets, $\{C_i\}_{i=1}^{\infty}$, such that $C_i\subseteqq E$ and $\mu_i(E-C_i)<a_i$ $(i=1, 2, \cdots)$ satisfying $a_i>0$ and $\sum_{i=1}^{\infty}a_i<\infty$. $\nu(E-\bigcup_{i=1}^{\infty}C_i)\leqq\sum_{i=1}^{\infty}\mu_i(E-C_i)<\sum_{i=1}^{\infty}a_i<\infty$ implies $\nu(\bigcup_{i=1}^{\infty}C_i)=\infty$, hence the existence of an integer N such that $\nu(\bigcup_{i=1}^{N}C_i)>M$ for an arbitrary $M>0$.

Thus, we obtain the inner regularity of E with respect to ν.

(3) Let a set $E\in S$ and arbitrary $\varepsilon>0$ be specified.

I. $\nu(E)<\infty$. There exist a sequence, $\{\lambda_i\}_{i=1}^{\infty}$, and a partition $\{A_i\}_{i=1}^{\infty}$ of E such that $\lambda_i\in\Lambda$ $(i=1, 2, \cdots)$, $\bigcup_{i=1}^{\infty}A_i=E$, $A_j\frown A_k=\theta$ $(j\neq k)$, $A_i\in S$ $(i=1, 2, \cdots)$ and $\nu(E)\geqq\mu_{\lambda_1}(A_1)+\mu_{\lambda_2}(A_2)+\cdots+\mu_{\lambda_i}(A_i)+\cdots>\nu(E)-\varepsilon/3$. By the assumption of the inner regularities of $\mu_\lambda(\lambda\in\Lambda)$, there exists a sequence of compact measurable sets, $\{C_i\}_{i=1}^{\infty}$, such that $C_i\subseteqq A_i$ and $\mu_{\lambda_i}(C_i)>\mu_{\lambda_i}(A_i)-\varepsilon/2^i\cdot 3$ $(i=1, 2, \cdots)$. Then, $\nu(\bigcup_{i=1}^{\infty}C_i)=\sum_{i=1}^{\infty}\nu(C_i)\geqq\sum_{i=1}^{\infty}\mu_{\lambda_i}(C_i)>\sum_{i=1}^{\infty}\mu_{\lambda_i}(A_i)-\varepsilon/3>\nu(E)-2\varepsilon/3$, hence $\nu(\bigcup_{i=1}^{N}C_i)>\nu(E)-\varepsilon$ for a suitable integer N.

II. $\nu(E)=\infty$. If $\mu_{\lambda_0}(E)=\infty$ holds for at least one index $\lambda_0\in\Lambda$, there exists a compact measurable set $C\subseteqq E$ such that $\mu_{\lambda_0}(C)>M$ for an arbitrary $M>0$, hence $\nu(C)>M$.

On the other hand, if $\mu_\lambda(E)<\infty(\lambda\in\Lambda)$, there exist a sequence, $\{\lambda_i\}_{i=1}^{\infty}$, and a partition $\{A_i\}_{i=1}^{\infty}$ of E such that $\lambda_i\in\Lambda$ $(i=1, 2, \cdots)$, $\bigcup_{i=1}^{\infty}A_i=E$, $A_j\frown A_k=\theta$ $(j\neq k)$, $A_i\in S$ $(i=1, 2, \cdots)$ and $\mu_{\lambda_1}(A_1)+\mu_{\lambda_2}(A_2)+\cdots+\mu_{\lambda_i}(A_i)+\cdots>3M$ for an arbitrary $M>0$. Next, there exists a sequence of compact measurable sets, $\{C_i\}_{i=1}^{\infty}$, such that $C_i\subseteqq A_i$ and $\mu_{\lambda_i}(C_i)>\mu_{\lambda_i}(A_i)-M/2^i$. Therefore, it holds that $\nu(\bigcup_{i=1}^{\infty}C_i)=\sum_{i=1}^{\infty}\nu(C_i)\geqq\sum_{i=1}^{\infty}\mu_{\lambda_i}(C_i)>\sum_{i=1}^{\infty}\mu_{\lambda_i}(A_i)-M>2M$ and $\nu(\bigcup_{i=1}^{N}C_i)>M$ for a suitable integer N.

Theorem 4. (1) Let $\nu=\bigcap_{\lambda\in\Lambda}\mu_\lambda$ be the inferior measure of $\{\mu_\lambda\}_{\lambda\in\Lambda}$. Then, if every μ_λ, $\lambda\in\Lambda$ is outer regular, ν is also outer regular.

(2) Let μ_1, μ_2 and ν be the three measures as follows (beforestated in this section): $\mu_1=\int f_1d\mu$, $\mu_2=\int f_2d\mu$, $\nu=\int(f_1\frown f_2)d\mu$. Then, if μ_1 and μ_2 are inner regular, then ν is also inner regular.

Proof. (1) We need consider only the case of ν-finite E. By the construction of the inferior measure, there exist a sequence, $\{\lambda_i\}_{i=1}^{\infty}$, and a partition $\{A_i\}_{i=1}^{\infty}$ of E such that $\lambda_i\in\Lambda$ $(i=1, 2,\cdots)$, $\bigcup_{i=1}^{\infty}A_i=E$, $A_j\frown A_k=\theta$ $(j\neq k)$, $A_i\in S$ $(i=1, 2, \cdots)$ and $\nu(E)\leqq\mu_{\lambda_1}(A_1)+\mu_{\lambda_2}(A_2)+\cdots$

$+\mu_{\lambda_i}(A_i)+\cdots<\infty$. Corresponding to a given $\varepsilon>0$, let U_i be an open measurable set containing A_i such that $\mu_{\lambda_i}(U_i)<\mu_{\lambda_i}(A_i)+\varepsilon/2^i$ $(i=1, 2, \cdots)$ and let $U=\bigcup_{i=1}^{\infty}U_i$. Then, U is measurable and open, containing E, and $\nu(U-E)\leqq\nu(U_1-E)+\nu(U_2-E)+\cdots+\nu(U_i-E)+\cdots\leqq\mu_{\lambda_1}(U_1-A_1)+\mu_{\lambda_2}(U_2-A_2)+\cdots+\mu_{\lambda_i}(U_i-A_i)+\cdots<\varepsilon$. Since $\varepsilon>0$ is otherwise arbitrary, the outer regularity of E with respect to ν is established.

(2) Generally, denote the sets $\{x: f_1(x)\leqq f_2(x)\}$ and $\{x: f_1(x)>f_2(x)\}$ by the symbols X_1 and X_2, respectively. Then, for a measurable set E, $\nu(E)=\nu(E\frown X_1)+\nu(E\frown X_2)=\int_{X_1\frown E}(f_1\frown f_2)d\mu+\int_{E\frown X_2}(f_1\frown f_2)d\mu=\int_{E\frown X_1}f_1d\mu+\int_{E\frown X_2}f_2d\mu=\mu_1(E\frown X_1)+\mu_2(E\frown X_2)$.

If $\nu(E)<\infty$, there exist the two compact measurable sets C_1 and C_2 such that $C_1\subseteq E\frown X_1$, $C_2\subseteq E\frown X_2$, $\mu_1(C_1)>\mu_1(E\frown X_1)-\varepsilon/2$ and $\mu_2(C_2)>\mu_2(E\frown X_2)-\varepsilon/2$, hence $\nu(E-(C_1\smile C_2))=\mu_1((E\frown X_1)-C_1)+\mu_2((E\frown X_2)-C_2)<\varepsilon$.

On the other hand, if $\nu(E)=\infty$, either $\mu_1(E\frown X_1)=\infty$ or $\mu_2(E\frown X_2)=\infty$. Suppose $\mu_1(E\frown X_1)=\infty$ for the present. Then, there exists a compact measurable set C such that $C\subseteq E\frown X_1$ and $\mu_1(C)>M$ for an arbitrary $M>0$, hence $\nu(C)=\mu_1(C)>M$.

Thus, ν is also inner regular, and (2) of the above theorem is proved.

References

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