135. *Mappings and Pseudo-compact Spaces*

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Let f be a mapping of a topological space X onto another topological space Y: then, by Whyburn [1, 2], Stone [3], Morita [4–6], Hanai [6–8], McDougle [9, 10] and others, it is known that some properties of f, for instance closedness, openness and quasi-compactness, give the interest relations between X and Y.

In this paper, we shall first prove that a space X is pseudo-compact if and only if any continuous mapping of X onto a weakly separable T_2-space is always a P_0-mapping. Next we shall show, for a continuous mapping f of a pseudo-compact space X onto a weakly separable T_2-space, that 1) f is quasi-compact if and only if $f(\mathfrak{B}U)=\mathfrak{B}f(U)$ for any open inverse subset U where $\mathfrak{B}U$ denotes the boundary of U, and 2) if $\mathfrak{B}f^{-1}(y)$ is compact for every $y\in Y$ and X is locally compact, then f is always closed and Y is locally compact. Finally we give some characterization of compact spaces.

In the following, we assume that any mapping is always continuous. Let f be a mapping of X onto Y where X and Y are topological spaces; f is a P_1 (or P_0)-*mapping* provided that whenever $y\in Y$ and U is any neighborhood of $f^{-1}(y)$, $y\in \operatorname{Int} f(U)$ (or $y\in \operatorname{Int} f(\overline{U})$). f is a P_2-*mapping* if for each $y\in Y$, there is a compact subset C of $f^{-1}(y)$ such that $\operatorname{Int} f(U)\ni y$ for every open subset U containing C (the definitions of both P_1 and P_2-mappings are due to McDougle [9]). f is called to have a *compact trace property* [2] if any point y of Y is interior of the image of some compact subset of X. The following implications are obvious: (*open* $\to P_1$), (*closed* $\to P_1\to$ *quasi-compact*) and ($P_2\to P_1\to P_0$).

1. Characterizations of pseudo-compact spaces. The following lemma is useful.

Lemma 1. *Let f be a mapping of a topological space X onto a T_2-space Y. If $\{y_n\}\to y$ in Y and x_n is any point contained in $f^{-1}(y_n)$, then $\overline{\{x_n\}}-\{x_n\}\ni x$ implies $f(x)=y$.*

Theorem 1. *The following conditions are equivalent for a complete regular T_1-space X;*

1) *X is pseudo-compact.*
2) *Any mapping of X onto a weakly separable T_2-space is P_0.*
3) *If f is a mapping of X onto a weakly separable T_2-space Y, then*

$\overline{f(U)}-f(U)\subset f(\mathfrak{B}U)$ for any open subset U of X.

4) *Under the same assumption as in* (3), *$\overline{f(\overline{U})}=f(\overline{U})$ for any open subset U of X.*

5) *Under the same condition as in* (3), *if a regular open subset U of X contains $f^{-1}(y)$, then* $\operatorname{Int} f(U)\ni y$.

Proof. ($1\to2$) Let $y\in Y$ and U be any open set containing $f^{-1}(y)$. Suppose that $\operatorname{Int} f(\overline{U})\not\ni y$. Since U contains $f^{-1}(y)$, there is a sequence $\{y_n\}$ in $Y-f(\overline{U})$ which converges to y. If $f^{-1}(y_n)-\overline{U}\ni x_n$, $\{x_n\}$ has no cluster points. For if x is a cluster point of $\{x_n\}$, $U\supset f^{-1}(y)$ implies $f^{-1}(y)\not\ni x$ and hence $\{f(x_n)(=y_n)\}$ converges to $f(x)(\neq y)$ which is impossible by Lemma 1. Let $\{V_n\}$ be a base of neighborhoods at the point y such that $V_n\supset\overline{V}_{n+1}$ and $V_n\ni y_m$ for all $m\geqslant n$. Since $\{x_n\}$ is closed and X is regular, we can select a family $\{U_n\}$ of open sets such that $U_n\ni x_n$, $\overline{U}_n\frown\overline{U}_m=\theta$ $(n\neq m)$, $\overline{U}_n\frown\overline{U}=\theta$ and $f(\overline{U}_n)\subset V_n$. Then the pseudo-compactness of X implies that $\{U_n\}$ is not locally finite (for instance, see [11]). Hence there is at least one point c in $\left(\overline{\bigcup_{n=1}^{\infty} U_n}\right)-\bigcup_{n=1}^{\infty} U_n$. By the method of construction of $\{U_n\}$, we have $f(c)=y$ which contradicts the fact that U is an open set containing $f^{-}(y)$. This shows that $f^{-1}(y_n)\subset\overline{U}$ and hence $y_n\in f(\overline{U})$ for every n. Thus y must be an interior point of $f(\overline{U})$.

($1\to3$) Suppose that $y\in\overline{f(U)}-f(U)$ and $f^{-1}(y)\subset X-\overline{U}$. Let $\{y_n\}$ be a sequence in $f(U)$, which converges to y and let $x_n\in f^{-1}(y_n)\frown U$. We select a base $\{V_n\}$ of neighborhoods at y such that $V_n\supset\overline{V}_{n+1}$. Then $\{x_n\}$ is closed because if $\overline{\{x_n\}}-\{x_n\}\ni c$, then $\overline{U}\ni c$ and $f(c)=y$, that is, $f^{-1}(y)\frown\overline{U}\neq\theta$ which is a contradiction. Let U_n be an open set containing x_n such that $\overline{U}_n\subset U\frown f^{-1}(V_n)$ and $\overline{U}_n\frown\overline{U}_m=\theta$ $(n\neq m)$. Since X is pseudo-compact, $\{U_n\}$ is not locally finite and hence there is a point c in $\overline{\{U_n\}}-\{U_n\}$. By the method of construction of $\{U_n\}$, $f(c)\in V_n$ for every n, and hence $f(c)=y$. On the other hand, the assumption that $f^{-1}(y)\subset X-\overline{U}$ implies $c\notin f^{-1}(y)$. This is a contradiction. Thus we have $\overline{f(U)}-f(U)\subset f(\overline{U}-U)=f(\mathfrak{B}U)$.

($1\to4$) This follows from ($1\to3$) and the fact that $f(\overline{U})\subset\overline{f(U)}$.

($1\to5$) Suppose that $\operatorname{Int} f(U)\not\ni y$. Then there is a sequence $\{y_n\}$ in $Y-f(U)$ which converges to y. As in the proof of ($1\to2$), $f^{-1}(y_n)\subset\overline{U}-U$, because $f^{-1}(y_n)\frown U=\theta$ for every n. If a point $x_n\in f^{-1}(y_n)$, except with finitely many n, has a neighborhood U_n such that $U_n\frown(X-\overline{U})\neq\theta$, then $\{U_n\frown(X-\overline{U})\}$ has no cluster points by Lemma 1. But this shows that $\{U_n\frown(X-\overline{U})\}$ is locally finite which contradicts the pseudo-compactness of X. Thus all points in $f^{-1}(y_n)$, except with finitely many n, have neighborhoods contained in U. This is impossible since U is a regular open subset of X.

(*Proofs of reverse implications*). Suppose that X is not pseudo-compact. Then there is a locally finite family $\{U_n\}$ of disjoint open sets of X. Let f be a non-negative continuous function such that $f\left(X-\bigcup_{n=1}^{\infty} U_n\right)=0$, $f(U_n)\leq n$ and $f(x_n)=n$ for some point $x_n\in U_n$. We construct a space Y, from $Z=f(X)$, with the following topology: a neighborhood U_n of the point 1 is the union of a neighborhood V_n of the point 1 in Z and a set $\{x; f(x)>n\}$ where $V_n=\{z; |z-1|<1/n\}$. Other points have the same neighborhoods as ones in Z. Let h be the identical mapping of Z onto Y. Then $g=hf$ is a mapping of X onto a weakly separable space Y.

Let $U=\{x; x\in X, f(x)<2\}$ and $V=\text{Int }(\bar{U})$. It is obvious that U is an open inverse set and V is a regular open subset containing $g^{-1}(1)$. By the methods of construction of Y, $g(U)$ does not contain the point 1 as an inner point. This shows that g is not P_0 which proves $(2\to 1)$. On the other hand, $V\supset g^{-1}(1)$ but Int $g(V)$ does not the point 1 as an inner point. This shows that $(5\to 1)$. As similarly, it is easy to see that $(3\to 1)$ and $(4\to 1)$.

2. Quasi-compact mappings and closed mappings. First we shall prove the following

Theorem 2. *Let f be a mapping of a pseudo-compact, completely regular T_1-space X onto a weakly separable T_2-space Y. f is quasi-compact if and only if $f(\mathfrak{B}U)=\mathfrak{B}f(U)$ for any open inverse subset of X.*

Proof. Suppose that f is quasi-compact and U is an open inverse set of X. By Theorem 1 and the openness of $f(U)$, we have $\mathfrak{B}f(U)=\overline{f(\bar{U})}-f(U)\subset f(\mathfrak{B}U)$. On the other hand, $f(\bar{U})\subset\overline{f(\bar{U})}$ and hence $f(\bar{U})-f(U)\subset\overline{f(\bar{U})}-f(U)$. Since U is an open inverse set, $f(\bar{U})=f(\mathfrak{B}U)\smile f(U)$, $f(\mathfrak{B}U)\frown f(U)=\theta$. Thus we have $f(\mathfrak{B}U)\subset\mathfrak{B}f(U)$ and hence we have $f(\mathfrak{B}U)=\mathfrak{B}f(U)$.

Conversely suppose that $f(\mathfrak{B}U)=\mathfrak{B}f(U)$ for any open inverse set U. Since U is an open inverse set, $\bar{U}=U\smile\mathfrak{B}U$, $U\frown\mathfrak{B}U=\theta$ and $f(U)\frown f(\mathfrak{B}U)=\theta$. Thus $f(\bar{U})$ is a union of two disjoint sets $f(U)$ and $f(\mathfrak{B}U)$. On the other hand, $\overline{f(U)}=\mathfrak{B}f(U)\smile\text{Int }f(U)$, $\mathfrak{B}f(U)\frown\text{Int }f(U)=\theta$. By Theorem 1, we have $f(\bar{U})=\overline{f(U)}$, and hence $f(U)=f(\bar{U})-f(\mathfrak{B}U)=\overline{f(U)}-f(\mathfrak{B}U)=\overline{f(U)}-\mathfrak{B}f(U)=\text{Int }f(U)$. This shows that $f(U)$ is open.

In §1 we proved that any mapping of a pseudo-compact space onto a weakly separable T_2-space is always a P_0-mapping. If X is not countably compact, then there is a mapping which is not closed (see §3 below or [11, Theorem 3]). K. Morita has proved that a quasi-compact mapping f of a semi-compact T_2-space X onto a T_2-space Y is closed if $f^{-1}(y)$ is connected and $\mathfrak{B}f^{-1}(y)$ is compact for every $y \in Y$ [5, Theorem 1 and Remarks]. McDougle [10, Lemma 2] has proved that if a mapping f of a topological space X onto a T_1-space Y is P_1 and if $\mathfrak{B}f^{-1}(y)$ is compact for each point $y \in Y$, then f is P_2. Whyburn [2] obtained the result that if X and Y are locally compact separable metric spaces, then $f(X)=Y$ has the property P_2 if and only if every compact set E in Y has a compact trace F, that is, $f(F)=E$. The proof of this result contains the following results: 1) if X is locally compact and $f(X)=Y$ is P_2, then any compact set in Y has a compact trace, and 2) if Y is locally compact and $f(X)=Y$ has the property such that any compact set in Y has a compact trace, then f is P_2. For a pseudo-compact space, we have the following

Theorem 3. *Let X be a pseudo-compact completely regular T_1-space and let f be a mapping of X onto a weakly separable T_2-space Y. Suppose that $\mathfrak{B}f^{-1}(y)$ is compact for every point $y \in Y$. Then,* 1) *f is always closed, and* 2) *if X is locally compact, then Y is locally compact and f is P_2 and moreover f has the compact trace property.*

Proof. 1) Let F be a closed subset of X and $y \in \overline{f(F)} - f(F)$. Then there is a sequence $\{y_n\}$ in $f(F)$, which converges to y. Let $x_n \in f^{-1}(y_n) \frown F$. Since $X-F$ contains $f^{-1}(y)$, $\{x_n\}$ is closed by Lemma 1. Since $\mathfrak{B}f^{-1}(y)$ is compact, there are disjoint open sets U and V such that $U \supset f^{-1}(y)$ and $V \supset F$. Let $\{V_n\}$ be a family of open sets such that $V_n \ni x_n$, $V \supset V_n$, $f(V_n) \subset W_n$ and $\overline{V}_n \frown \overline{V}_m = \theta$ $(n \neq m)$ where $\{W_n\}$ is a base of neighborhoods at y. Since X is pseudo-compact, $\{V_n\}$ is not locally finite and hence $\overline{\{V_n\}} - \{V_n\}$ contains a point c. It is easy to see that $f(c)=y$. But $c \notin f^{-1}(y)$ because $(\overline{\bigcup V_n}) \subset \overline{V} \subset X-U$. Thus f must be closed.

2) Let y be a point of Y. If $\mathfrak{B}f^{-1}(y)=\theta$, $f^{-1}(y)$ is open and closed. This leads that the point y is an isolated point in Y by the pseudo-compactness and weak separability of Y. Thus we lose no generality by assuming that $\mathfrak{B}f^{-1}(y)$ is not empty. (In the following proof, the pseudo-compactness of X and the weak separability of Y are not used and we use only a local compactness of X and a P_1-property of mapping.) To prove the theorem, it suffices to show that Y is locally compact. Since $\mathfrak{B}f^{-1}(y)$ is compact, we can select an open covering $\{U_1, \cdots, U_n\}$ of $\mathfrak{B}f^{-1}(y)$ such that $\overline{U}_i$ is compact. Thus we have that $A = \mathrm{Int}\left(\bigcup_{i=1}^{n} U_i\right) \smile \mathrm{Int}\ (f^{-1}(y)) \ni y$. Since f is P_1 and $f(A)$

$\subset f\left(\bigcup_{i=1}^{n} U_i\right)$, we have that $y \in \text{Int } f(A) \subset \text{Int } f\left(\bigcup_{i=1}^{n} U_i\right) \subset \text{Int } f\left(\bigcup_{i=1}^{n} \bar{U}_i\right)$. This shows that y has an arbitrary small compact neighborhoods. Thus Y is locally compact.

3. Countably compact spaces and compact spaces. In §1 we obtained a characterization of pseudo-compact spaces. In this section, we shall consider, in the same direction as in §1, a characterization of compact spaces. For countably compact spaces, we have obtained the following theorem [11, Theorem 3]:

a completely regular T_1-space X is countably compact if and only if any mapping of X onto a weakly separable T_2-space is always closed.

For compact spaces we have

Theorem 4. *The following conditions are equivalent for a completely regular T_1-space X:*

1) *X is compact;*
2) *any mapping of X onto a weakly separable T_2-space Y is compact;*
3) *any mapping of X onto a completely regular T_1-space is P_0;*
4) *any mapping of X onto a completely regular T_1-space is quasi-compact.*

Proof. $(1\to 2)$ and $(1\to 4\to 3)$ are obvious.

$(2\to 1)$ Suppose that X is not compact, that is, there is a point x^* in $\beta X - X$. Let f be a continuous function on βX which vanishes on some neighborhood of x^*. If we put $h = f \mid X$, then $h^{-1}(0)$ is not compact.

$(3\to 1)$ Suppose that X is not compact and Y is a space obtained from $X \cup \{x^*\}$ by contracting x^* to a fixed point x in X where $x^* \in \beta X - X$. Then Y is a one-to-one continuous image of X, we denote by f this mapping. Let U be an open set, in X, containing x such that $\bar{U}(\text{in } \beta X) \not\ni x^*$. It is obvious that $\text{Int } f(U) \not\ni x$. This shows that f is not P_0.

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