7. A Class of Quasi-normed Spaces

By Kiyoshi ISÉKI

(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1960)

A non-Archimedean normed space was considered by A. F. Monna [2] and I. S. Cohen [1]. We shall consider a non-Archimedean quasi-normed space. By a non-Archimedean quasi-normed space with power r, we shall mean a linear space E over a commutative field K such that to every x of E there corresponds a real number $||x||$ such that

1) $||x||>0$ for $x\neq 0$.

2) $||x+y||\leq \text{Max}(||x||, ||y||)$ for x, y of E.

3) $||\lambda x||=|\lambda|^r ||x||$ for $\lambda\in K$ and $x\in E$,

where $|\lambda|$ is a non-Archimedean valuation of K.

It is clear that the function $d(x, y)=||x-y||$ defines a metric on the space E.

Now we shall show the following

Proposition 1. *Let F be a closed linear subspace of E. If the sequence $\{x_n+a_n y\}$ converges in E for a fixed element $y\bar{\in} F$ and $x_n\in F$, $a_n\in K$, then $\{x_n\}$ and $\{a_n\}$ are convergent.*

Proof. We shall prove that $x_n+a_n y\to 0$ $(n\to\infty)$ implies $a_n\to 0$. Suppose that a_n does not converge, then there is a subsequence $\{a_{k_n}\}$ such that $|a_{k_n}|\geq\varepsilon$ for some positive number ε. Hence

$$||a_{k_n}^{-1}(x_{k_n}+a_{k_n}y)=|a_{k_n}^{-1}|^r ||x_{k_n}+a_{k_n}y||$$
$$\leq \varepsilon^{-r}||x_{k_n}+a_{k_n}y||$$

implies $a_{k_n}^{-1}(x_{k_n}+a_{k_n}y)=a_{k_n}^{-1}x_{k_n}+y\to 0$. From $a_{k_n}^{-}x_{k_n}\in F$ and closedness of F, we have $y\in F$.

For the general case, from the existence of $\lim_{n\to\infty}(x_n+a_n y)$, we have $(x_{n+1}-x_n)+(a_{n+1}-a_n)y\to 0$, and we can conclude $a_{n+1}-a_n\to 0$. Hence, by a property of non-Archimedean valuation $\{a_n\}$ is a fundamental sequence and we can find the limit of $\{a_n\}$.

Proposition 2. *Any finite dimensional subspace of E is closed.*

Proof. If F is a closed linear subspace of E, then $F+[y]$ is closed, where $F+[y]$ denotes the minimal linear subspace generated by F and y, i.e. $F+[y]=\{x+\alpha y, x\in F, \alpha\in K\}$. Suppose that $y\bar{\in} F$, and $\{x_n+\alpha_n y\}(x_n\in F, \alpha_n\in K)$ converges to an element of E, then, by Proposition 1, x_n converges to an element x of E, and α_n converges to α of E. Since F is closed, the element x is contained in F. Therefore $x+\alpha y \in F+[y]$, and $F+[y]$ is closed.

Theorem 1. *Any finite dimensional non-Archimedean normed space E is topologically Euclidean and is complete.*

Proof. Let e_i $(i=1, 2, \cdots, n)$ be a basis for E. Any element is represented as the form $\sum_{i=1}^{n} \alpha_i e_i$, $\alpha_i \in K$. Therefore

$$\left\|\sum_{i=1}^{n} \alpha_i e_i\right\| \leq \operatorname*{Max}_{1 \leq i \leq n} \|\alpha_i e_i\| = \operatorname*{Max}_{1 \leq i \leq n} |\alpha_i|^{\lambda} \|e_i\|.$$

Let $M = \operatorname*{Max}_{1 \leq i \leq n} |\alpha_i|^{\lambda}$, then

$$\left\|\sum_{i=1}^{n} \alpha_i e_i\right\| \leq M \operatorname*{Max}_{1 \leq i \leq n} \|e_i\|.$$

Hence the mapping $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \to \sum_{i=1}^{n} \alpha_i x_i$ is continuous. Conversely, by Proposition 2, the linear subspace $[e_1] + \cdots + [e_{n-1}]$ is closed, and the linear subspace does not contain e_n. Hence if $\sum_{i=1}^{n} \alpha_i^{(k)} e_i \to 0$ $(k \to \infty)$ from Proposition 1, we have $\alpha_n^{(k)} \to 0$ $(k \to \infty)$.

Similarly, we have $\alpha_i^{(k)} \to 0$ $(k \to \infty)$ for every i. This completes the proof.

References

[1] I. S. Cohen: On non-Archimedean normed spaces, Indagationes Math., **10**, 244-249 (1948).

[2] A. F. Monna: Sur les espaces linéaires normés I-IV, Proc. Kon. Ned. Akad. Wetensch. Amsterdam, **49** (1946).