5. A Remark on Quasi-Frobenius Rings

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1. Throughout this note A will denote a ring with unit satisfying the minimum conditions for left and right ideals.

We shall consider the following conditions:

Condition (P). Let M be a module. If two submodules N_1 and N_2 are isomorphic, then the two residue modules M/N_1 and M/N_2 are isomorphic, too.

Condition (L_n). Let A be a ring and $A^{(n)}$ the direct sum of n isomorphic copies of the left A-module A. Then the module $A^{(n)}$ satisfies (P).

Condition (R_n)=the right-left symmetry of (L_n).

The following facts are known:

(1) If A is quasi-Frobenius (in short, QF), then A satisfies (L_n) and (R_n) for $n=1, 2, \cdots$ [3, Corollary 4.3]. (See also [4, Theorem 2.3].)

(2) Conversely, if A satisfies (L_n) and (R_n) for every natural number n, then A is QF [3, Theorem 4.4].

(3) If A is an algebra over an algebraically closed field, and if A satisfies (L_1) (or (R_1)), then A is QF [2, Theorem 3].

(4) There exists an algebra A satisfying (L_1) and (R_1) which is not QF [2, Remark].

Lemma 1. Let M be a module of finite length, and D a direct summand of M. If M satisfies (P), D also satisfies (P).

Proof. Let N_1 and N_2 be mutually isomorphic submodules of D, and let $M=D\oplus D'$. Then, $(D/N_1)\oplus D'\simeq M/N_1\simeq M/N_2\simeq (D/N_2)\oplus D'$ by assumption. Therefore $D/N_1\simeq D/N_2$ by the Krull-Remak-Schmidt theorem, as desired.

From this lemma and the proof of [3, Theorem 4.4] it follows that every ring A satisfying (L_2) *and* (R_2) is QF. The purpose of the present note is to show the following

Theorem 2. *Let A be a ring, and B a left A-module. Suppose that* (1) *A is a direct summand of B, and* (2) *for every indecomposable summand Ae_k of A, B contains a direct summand which is the direct sum of two isomorphic copies of Ae_k. If B satisfies* (P), *then A is* QF.

As immediate consequences we obtain

Corollary 3. If A satisfies (L_2) (*or* (R_2)) then A is QF.

Corollary 4. Let A_2 be the total matrix ring of degree 2 over

A. If the left (or right) A_2-module A_2 satisfies (P), A is QF.

Corollary 5. Let every simple summand of A modulo radical have the capacity >1. If the left (or right) A-module A satisfies (P), then A is QF.

2. In order to prove Theorem 2 we note first the following

Lemma 6. A ring A is QF whenever the following conditions are satisfied:

(i) If two left ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are isomorphic, then $\mathfrak{l}_2=\mathfrak{l}_1 a$ for some $a\in A$.

(ii) Let e_1 and e_2 be primitive idempotents, and let $\mathfrak{l}_1$ be a left ideal contained in Ae_1. If a homomorphism v of $\mathfrak{l}_1$ into Ae_2 is not one-to-one, there is an element a such that v is given by the right multiplication of a.

(iii) Let e be a primitive idempotent, and suppose that Ae is subdirectly irreducible. Let $\mathfrak{l}_1$ be a subideal of Ae. Then every homomorphism v of $\mathfrak{l}_1$ into Ae is given by the right multiplication of an element of A.

Proof. It is easy to verify that the proofs of [1, Lemma 2] and [1, Proposition 2] are still valid literally under our much weaker assumption, except the first part of the proof of [1, Proposition 2] which shows that $r(N)e_k$ is simple. However, the simplicity is easily proved in the following way. We assume that the conclusion of [1, Lemma 2] already has been verified. Let us suppose that $r(N)e_k$ is a direct sum of mutually isomorphic simple left ideals $\mathfrak{m}_j$, $j=1,\cdots,s$. Let $s>1$. Denote an isomorphism of $\mathfrak{m}_j$ onto $\mathfrak{m}_{j+1}$ by w_j for $j=1,\cdots,s-1$. We consider the endomorphism of $r(N)e_k$ which coincides with w_j on $\mathfrak{m}_j$ for each $j=1,\cdots,s-1$, and maps $\mathfrak{m}_s$ to 0. By virtue of our assumption (ii) this endomorphism is given by the right multiplication of an element a. Evidently we may assume that $a\in e_kAe_k$. Since $a^s=0$, $a\in e_kNe_k\subseteq N$. Therefore, $(r(N)e_k)a\subseteq l(N)e_ka\subseteq l(N)N=0$, a contradiction. Thus, $s=1$ and $r(N)e_k$ is simple, as desired.

Lemma 7. Let M be a module and u an automorphism of M. Suppose that $M=M_1\oplus M_2$, where M_1 is an indecomposable submodule of finite length. Then $u(M_1)\cap M_1=0$ or $u(M_1)\cap M_2=0$.

Proof. Denote the projections of $M=M_1\oplus M_2$ on M_1 and M_2 by p_1 and p_2 respectively. Then, evidently $up_1u^{-1}=up_1u^{-1}p_1+up_1u^{-1}p_2$. It follows that $up_1u^{-1}p_1$ or $up_1u^{-1}p_2$ gives an automorphism of $u(M_1)$. Therefore, either $u(M_1)\cap M_2\subseteq u(M_1)\cap \operatorname{Ker}(up_1u^{-1}p_1)=0$ or $u(M_1)\cap M_1 \subseteq u(M_1)\cap \operatorname{Ker}(up_1u^{-1}p_2)=0$, as desired.

Proof of Theorem 2. It suffices to verify the conditions (i)–(iii) in Lemma 6.

(a) If two left ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ of A are isomorphic, then $A/\mathfrak{l}_1\simeq A/\mathfrak{l}_2$ by assumption and Lemma 1. In view of [4, Theorem 1.3] there is an element a such that the right multiplication of a induces the isomorphism of $A/\mathfrak{l}_1$ and $A/\mathfrak{l}_2$. Thus $\mathfrak{l}_1 a=\mathfrak{l}_2$.

(b) Let e_1 and e_2 be primitive idempotents and v a homomorphism of a subideal $\mathfrak{l}_1$ of Ae_1 into Ae_2. By assumption, B contains a direct summand $M=M_1\oplus M_2$ which is a direct sum of submodules M_i isomorphic to Ae_i, $i=1,2$. Denote the isomorphism of Ae_i onto M_i by q_i, and set $N_1=q_1(\mathfrak{l}_1)$. Then, $q_2vq_1^{-1}$ gives a homomorphism w of N_1 into M_2. Now we denote by N the module consisting of all $x+w(x)$ for $x\in N_1$. Evidently $N\simeq N_1$. Hence $B/N\simeq B/N_1$ by assumption, and so $M/N\simeq M/N_1$ by Lemma 1. Thus, there exists an automorphism u of M such that $u(N_1)=N$ by [4, Theorem 1.3]. Let us suppose that $u(M_1)\cap M_2=0$. Then we have $u(M_1)\oplus M_2\simeq M_1\oplus M_2=M$, and so $u(M_1)\oplus M_2=M$. Let $q_1(e_1)=x+y$, $x\in u(M_1)$, $y\in M_2$, and let $z\in\mathfrak{l}_1$. $M_2\ni wq_1(z)+zy=(q_1(z)+wq_1(z))-z(q_1(e_1)-y)=(q_1(z)+wq_1(z))-zx\in N+u(M_1)=u(N_1)+u(M_1)=u(M_1)$. Hence $wq_1(z)+zy\in u(M_1)\cap M_2=0$. Therefore $q_2v(z)=wq_1(z)=-zy=q_2(z(q_2^{-1}(-y)))$, and so $v(z)=z(q_2^{-1}(-y))$ for every $z\in\mathfrak{l}_1$.

(c) To verify (ii) we assume that v is not one-to-one. Then $0\neq q_1(\operatorname{Ker} v)=\operatorname{Ker} w=N\cap M_1=u(N_1)\cap M_1\subseteq u(M_1)\cap M_1$, whence $u(M_1)\cap M_2=0$ by Lemma 7. Therefore from (b) it follows that v is given by the right multiplication of $q_2^{-1}(-y)\in A$.

(d) Finally, in order to verify (iii) we suppose that $v\neq 0$ and that $Ae=Ae_1=Ae_2$ is subdirectly irreducible. By virtue of the argument (b) we need only to show that $u(M_1)\cap M_2=0$. Now let $u(M_1)\cap M_2\neq 0$. Then $u(M_1)\cap M_1=0$ by Lemma 7. For any $w(t)\in u(M_1)\cap w(N_1)$ we have $t+w(t)\in N=u(N_1)\subseteq u(M_1)$. Hence $t\in u(M_1)$ and $t\in u(M_1)\cap N_1\subseteq u(M_1)\cap M_1=0$, whence $t=0$. Thus $u(M_1)\cap w(N_1)=0$, and so $u(M_1)\cap(M_2\cap q_2v(\mathfrak{l}_1))=u(M_1)\cap q_2v(\mathfrak{l}_1)=u(M_1)\cap w(N_1)=0$. Therefore $(u(M_1)\cap M_2)\cap q_2v(\mathfrak{l}_1)=0$, which contradicts the subdirect irreducibility of $M_2=q_2(Ae)$, completing the proof.

References

[1] M. Ikeda: A characterization of quasi-Forbenius rings, Osaka Math. J., **4**, 203-209 (1952).
[2] Yutaka Kawada: On similarities and isomorphisms of ideals in a ring, J. Math. Soc. Japan, **9**, 374-380 (1957).
[3] K. Morita and H. Tachikawa: Character modules, submodules of a free module, and quasi-Frobenius rings, Math. Zeitschr., **65**, 414-428 (1956).
[4] K. Morita, Yutaka Kawada, and H. Tachikawa: On injective modules, Math. Zeitschr., **68**, 217-226 (1957).