132. *Perturbation Theory of the Electromagnetic Fields in Anisotropic Inhomogeneous Media*

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Since it is very difficult and complicated to analyse the fields in anisotropic inhomogeneous media exactly, some approximated methods have been studied, but it seems to the author that few of them are rigorous and general enough. In this paper he will develop the perturbation theory of the fields which is not only simple but rigorous and general under one assumption that the deviations of the anisotropicity and inhomogeneity are not so large.

1. Suppose that the properties of the medium are represented as $\boldsymbol{D}=[\varepsilon]\boldsymbol{E}+[\xi]\boldsymbol{H}$, $\boldsymbol{B}=[\mu]\boldsymbol{H}+[\zeta]\boldsymbol{E}$ and $\boldsymbol{K}=[\sigma]\boldsymbol{E}$, where $[\varepsilon]$ etc. are 3×3 matrices, the elements ε_{ij} $(i, j=1, 2, 3)$ of which are functions of position. Let $\varepsilon'_{ij}=\varepsilon_{ij}-\varepsilon\delta_{ij}$ where ε is a properly chosen constant, then $[\varepsilon]=\varepsilon U+[\varepsilon']$, where U is the unit matrix and $[\varepsilon']$ is a matrix with elements ε'_{ij}. Similar holds for $[\mu]$ and $[\sigma]$, and Maxwell's equation will be

(1) $$\nabla\times\boldsymbol{E}=-i\omega\mu\boldsymbol{H}-\boldsymbol{K}_H,\qquad \nabla\times\boldsymbol{H}=(\sigma+i\omega\varepsilon)\boldsymbol{E}+\boldsymbol{K}_E$$

where $\boldsymbol{K}_H=i\omega([\zeta]\boldsymbol{E}+[\mu']\boldsymbol{H})$ and $\boldsymbol{K}_E=[\sigma'+i\omega\varepsilon']\boldsymbol{E}+i\omega[\xi]\boldsymbol{H}$. (1) represents the fields in isotropic and homogeneous medium with medium constants ε, μ and σ, in the presence of the distributions of densities of electromagnetic currents $\boldsymbol{K}_E$ and $\boldsymbol{K}_H$.

Assume that $\bar{\varepsilon}=\frac{1}{3}\sum_n\varepsilon_{nn}$ and $\varepsilon=\int\bar{\varepsilon}dV\Big/\int dV$, i.e. $\bar{\varepsilon}$ is the mean value of the diagonal elements of $[\varepsilon]$, which may still be a function of position, and ε is a constant which is the mean value of $\bar{\varepsilon}$ in the domain with which we are concerned. Constants μ and σ will be obtained in a similar way. With these constants we shall assume that

(2) $$\left|\frac{\varepsilon'_{mn}}{\varepsilon}\right|,\ \left|\frac{\xi_{mn}}{\varepsilon}\right|,\ \left|\frac{\mu'_{mn}}{\mu}\right|,\ \left|\frac{\zeta_{mn}}{\mu}\right|,\ \left|\frac{\sigma'_{mn}}{\sigma}\right|=0(\epsilon)$$

where ϵ is a number less than 1 and $0(\epsilon)$ is the Landau symbol. Under this assumption, it is easy to see that $\boldsymbol{K}_E$ and $\boldsymbol{K}_H$ are quantities of $0(\epsilon)$, and that when $\epsilon\to 0$, (1) reduces to

(3) $$\nabla\times\boldsymbol{E}_0=-i\omega\mu\boldsymbol{H}_0,\qquad \nabla\times\boldsymbol{H}_0=(\sigma+i\omega\varepsilon)\boldsymbol{E}_0$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are the fields in isotropic and homogeneous medium with medium constants ε, μ and σ. Therefore if we put as $\boldsymbol{E}=\boldsymbol{E}_0+\boldsymbol{e}$, and $\boldsymbol{H}=\boldsymbol{H}_0+\boldsymbol{h}$, then $\boldsymbol{e}$ and $\boldsymbol{h}$ are the deviations of $\boldsymbol{E}$ and $\boldsymbol{H}$ from $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ respectively, which resulted from the anisotropicity and inhomogeneity of the medium. It is easy to see that $\boldsymbol{e}$ and $\boldsymbol{h}$ are quantities of $0(\epsilon)$, hence $[\mu']\boldsymbol{h}$, $[\xi]\boldsymbol{h}$, $[\sigma'+i\omega\varepsilon']\boldsymbol{e}$ and $[\zeta]\boldsymbol{e}$ are quantities of $0(\epsilon^2)$. Thus neglecting quantities of $0(\epsilon^2)$, (1) reduces to

(4) $$\nabla\times\boldsymbol{E}=-i\omega\mu\boldsymbol{H}-\boldsymbol{k}_H,\quad \nabla\times\boldsymbol{H}=(\sigma+i\omega\varepsilon)\boldsymbol{E}+\boldsymbol{k}_E$$

where $\boldsymbol{k}_H=i\omega([\zeta]\boldsymbol{E}_0+[\mu']\boldsymbol{H}_0)$ and $\boldsymbol{k}_E=[\sigma'+i\omega\varepsilon']\boldsymbol{E}_0+i\omega[\xi]\boldsymbol{H}_0$.

Since $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are unperturbed fields in isotropic and homogeneous medium, they are solvable by the customary ways, hence they may be considered to be known functions. Therefore $\boldsymbol{k}_E$ and $\boldsymbol{k}_H$ also are known functions.

Thus we have shown that the fields in anisotropic inhomogeneous medium are equivalent, when terms of $0(\epsilon^2)$ are neglected, to the forced oscillations in isotropic and homogeneous medium in the presence of known densities of electromagnetic currents $\boldsymbol{k}_E$ and $\boldsymbol{k}_H$. The latter will be solvable with the applications of customary ways, of which we shall study in the following paper.

2. Although ω has been treated as if it was a known constant in the preceding section, the value of it is not known yet. Being an eigen value of (1) in anisotropic inhomogeneous medium, ω may have some deviation $\Delta\omega$ from the proper angular velocity ω_0 in isotropic and homogeneous medium, i.e. $\omega=\omega_0+\Delta\omega$. If we put as $\boldsymbol{E}(\omega)=\boldsymbol{E}_0(\omega_0)+\boldsymbol{e}(\omega)$, $\boldsymbol{H}(\omega)=\boldsymbol{H}_0(\omega_0)+\boldsymbol{h}(\omega)$ and $\omega=\omega_0+\Delta\omega$, then we shall have easily

(5) $$\begin{aligned}\nabla\times\boldsymbol{e}(\omega)&=-i\omega_0\mu\boldsymbol{h}(\omega)-i\mu\boldsymbol{H}_0(\omega_0)\cdot\Delta\omega-\boldsymbol{k}_H(\omega_0)\\ \nabla\times\boldsymbol{h}(\omega)&=(\sigma+i\omega_0\varepsilon)\boldsymbol{e}(\omega)+i\varepsilon\boldsymbol{E}_0(\omega_0)\cdot\Delta\omega+\boldsymbol{k}_E(\omega_0)\end{aligned}$$

where $\boldsymbol{k}_H(\omega_0)$ and $\boldsymbol{k}_E(\omega_0)$ are the values of $\boldsymbol{k}_H$ and $\boldsymbol{k}_E$, in which ω is replaced by ω_0.

$\boldsymbol{e}(\omega)$ and $\boldsymbol{h}(\omega)$ have been obtained in the preceding section. Suppose that $\boldsymbol{e}(\omega_0)$ and $\boldsymbol{h}(\omega_0)$ be the values of $\boldsymbol{e}(\omega)$ and $\boldsymbol{h}(\omega)$ respectively, in which ω is replaced by ω_0. It is easy to see that $\boldsymbol{e}(\omega)-\boldsymbol{e}(\omega_0)$ and $\boldsymbol{h}(\omega)-\boldsymbol{h}(\omega_0)$ are quantities of $0(\epsilon^2)$. Hence we can substitute $\boldsymbol{e}(\omega_0)$ and $\boldsymbol{h}(\omega_0)$ into (5) in place of $\boldsymbol{e}(\omega)$ and $\boldsymbol{h}(\omega)$ respectively. Then (5) will give a linear equation to determine $\Delta\omega$, for instance, if $\{\ \}_n$ be n-th component of the vector in $\{\ \}$, $\Delta\omega$ will be given as

(6) $$\Delta\omega=-\{\nabla\times\boldsymbol{e}(\omega_0)+i\omega_0\mu\boldsymbol{h}(\omega_0)+\boldsymbol{k}_H(\omega_0)\}_n/\{i\mu\boldsymbol{H}_0(\omega_0)\}_n.$$

Thus we have had the formula to determine the deviation of ω.

3. Appendix. We shall discuss about some properties of the fields here, though they do not come within the category of the perturbation theory.

It is well known that no density of space charge ρ can exist permanently in isotropic and homogeneous medium, but in our medium it is not necessary the case. It means physically that, being interrupted by the anisotropicity or inhomogeneity, some space charges can not fade away. Examples of the fields can be shown easily, in which $\rho \neq 0$, by the use of (1).

It is worth noting that the potentials have no use in general in the analysis of the fields in our case, because they can not be obtained independently of the fields.

There are no plane waves, in the sense of the old theory, in our medium generally. But if we think of the fields which depend on z and t only through the factor $e^{i\omega t+\gamma z}$, Maxwell's equation will be $\gamma \boldsymbol{i}_z \times \boldsymbol{E} = -i\omega[\mu]\boldsymbol{H} - i\omega[\zeta]\boldsymbol{E}$ and $\gamma \boldsymbol{i}_z \times \boldsymbol{H} = [\sigma + i\omega\varepsilon]\boldsymbol{E} + i\varepsilon[\xi]\boldsymbol{H}$. These will be unified in a matrix form as follows: $[\Gamma][{}^{H}_{E}]=[0]$, where $[\Gamma]$ is the coefficient matrix and

$$[\Gamma] \equiv \begin{bmatrix} i\omega\xi_{11}, & i\omega\xi_{12}+\gamma, & i\omega\xi_{13}, & (\sigma_{11}+i\omega\varepsilon_{11}), & (\sigma_{12}+i\omega\varepsilon_{12}), & (\sigma_{13}+i\omega\varepsilon_{13}) \\ i\omega\xi_{21}-\gamma, & i\omega\xi_{22}, & i\omega\xi_{23}, & (\sigma_{21}+i\omega\varepsilon_{21}), & (\sigma_{22}+i\omega\varepsilon_{22}), & (\sigma_{23}+i\omega\varepsilon_{23}) \\ i\omega\xi_{31}, & i\omega\xi_{32}, & i\omega\xi_{33}, & (\sigma_{31}+i\omega\varepsilon_{31}), & (\sigma_{32}+i\omega\varepsilon_{32}), & (\sigma_{33}+i\omega\varepsilon_{33}) \\ i\omega\mu_{11}, & i\omega\mu_{12}, & i\omega\mu_{13}, & i\omega\zeta_{11}, & i\omega\zeta_{12}-\gamma, & i\omega\zeta_{13} \\ i\omega\mu_{21}, & i\omega\mu_{22}, & i\omega\mu_{23}, & i\omega\zeta_{21}+\gamma, & i\omega\zeta_{22}, & i\omega\zeta_{23} \\ i\omega\mu_{31}, & i\omega\mu_{32}, & i\omega\mu_{33}, & i\omega\zeta_{31}, & i\omega\zeta_{32}, & i\omega\zeta_{33} \end{bmatrix}.$$

In order that there are non trivial solutions, we must have

$$(7) \qquad \Gamma \equiv Det. [\Gamma] = 0.$$

This is the characteristic equation for the propagation constant γ of 'plane wave'. Particularly, if

$[\sigma'] = [\xi] = [\zeta] = [0]$, $[\varepsilon] = \begin{bmatrix} \varepsilon_1, & i\varepsilon_2, & 0 \\ -i\varepsilon_2, & \varepsilon_1, & 0 \\ 0, & 0, & \varepsilon_3 \end{bmatrix}$ and $[\mu] = \begin{bmatrix} \mu_1, & i\mu_2, & 0 \\ -i\mu_2, & \mu_1, & 0 \\ 0, & 0, & \mu_3 \end{bmatrix}$

then (7) reduces to

$$(8) \quad \Gamma \equiv \gamma^4 - 2\{\omega^2\varepsilon_2\mu_2 + i\omega\mu_1(\sigma + i\omega\varepsilon_1)\}\gamma^2 - \omega^2(\mu_1^2 + \mu_2^2)\{(\sigma + i\omega\varepsilon_1)^2 - \omega^2\varepsilon_2^2\} = 0$$

which shows that there are two kinds of forward waves and two kinds of backward waves, which is well known in microwave engineering.