1. Note on Paracompactness

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1. Suggested by a well-known theorem of C. H. Dowker [1] that a topological space is countably paracompact and normal if and only if the product space $X\times I$ is normal, we have established the following theorem in a previous paper [2].

Theorem 1.1. *A topological space X is $\mathfrak{m}$-paracompact and normal if and only if the product space $X\times I^{\mathfrak{m}}$ is normal, where $\mathfrak{m}$ is an infinite cardinal number.*

Here a topological space X is called $\mathfrak{m}$-paracompact if any open covering of power $\leqq\mathfrak{m}$ admits a locally finite open refinement, and $I^{\mathfrak{m}}$ means the product space of $\mathfrak{m}$ copies of I, where $\mathfrak{m}$ is a cardinal number and I is the closed line interval [0, 1]. A topological space X is, by definition, paracompact if X is $\mathfrak{m}$-paracompact for any cardinal number $\mathfrak{m}$; furthermore, X is paracompact if X is $\mathfrak{m}$-paracompact for a cardinal number $\mathfrak{m}$ not less than the power of an open base of X. Accordingly, Theorem 1.1 gives a new characterization of paracompact spaces. Of course, "$\aleph_0$-paracompact" is nothing else "countably paracompact".

The purpose of this paper is to prove the following theorem which is a generalization of Theorem 1.1.

Theorem 1.2. *A topological space X is $\mathfrak{m}$-paracompact and normal if and only if the product space $X\times C^{\mathfrak{m}}$ is normal, where C is any compact metric space containing at least two points and $C^{\mathfrak{m}}$ means the product space of $\mathfrak{m}$ copies of C, and $\mathfrak{m}$ is an infinite cardinal number.*

As a special case where C is a space consisting of exactly two points we obtain the following theorem.

Theorem 1.3. *A topological space X is $\mathfrak{m}$-paracompact and normal if and only if the product space $X\times D^{\mathfrak{m}}$ is normal, where D is a discrete space consisting of two points and $D^{\mathfrak{m}}$ means the product space of $\mathfrak{m}$ copies of D, and $\mathfrak{m}$ is a cardinal number $\geqq 1$.*

The space $D^{\mathfrak{m}}$ is called a Cantor space, and $D^{\aleph_0}$ is the Cantor discontinuum.

It should be noted that in case $\mathfrak{m}=\aleph_0$, as far as the "if" part is concerned Theorem 1.3 gives a stronger form than Dowker's theorem while Theorem 1.1 gives a weaker form, and that for a finite cardinal number $\mathfrak{m}\geqq 1$, Theorem 1.3 is true but Theorem 1.1 is not.

2. We shall begin with a lemma concerning closed mappings.

Lemma 2.1. *Let f_i be a closed continuous mapping of a topological space X_i onto another topological space Y_i such that $f_i^{-1}(y)$ is compact for each point y of Y_i, $i=1,2$. If we put $g(x_1, x_2)=(f_1(x_1), f_2(x_2))$ for $x_i \in X_i$, $i=1,2$, then g is a closed continuous mapping of $X_1 \times X_2$ onto $Y_1 \times Y_2$.*

Proof. Let A be any closed subset of $X_1 \times X_2$. Suppose that $(y_1, y_2) \in \overline{g(A)}$. Then, for any open set H_i of Y_i such that $y_i \in H_i$ we have $(H_1 \times H_2) \frown g(A) \neq \phi$. Hence $(f_1^{-1}(H_1) \times f_2^{-1}(H_2)) \frown A \neq \phi$. Therefore we have $(f_1^{-1}(y_1) \times f_2^{-1}(y_2)) \frown A \neq \phi$; because, otherwise there would exist an open set G_1 of X_1 and an open set G_2 of X_2 such that $(G_1 \times G_2) \frown A = \phi$, $f_i^{-1}(y_i) \subset G_i$, $i=1,2$ since $f_i^{-1}(y_i)$ is compact for $i=1,2$, and we would have $(f_1^{-1}(L_1) \times f_2^{-1}(L_2)) \frown A = \phi$ where $L_i = Y_i - f_i(X_i - G_i)$, $i=1, 2$, since $f_i^{-1}(L_i) \subset G_i$ because of the closedness of f_i. Therefore $(y_1, y_2) \in g(A)$. This shows that g is a closed mapping.

Remark. If for at least one i, f_i does not satisfy the condition that $f_i^{-1}(y)$ be compact for each point y of Y_i, the closedness of the mapping g is not concluded in general. We shall give an example.

Let X_1 be the space of real numbers and Y_1 the quotient space obtained from X_1 by contracting the set of all integers to a point y_0; let f_1 be the identification map. Let $f_2: X_2 \to Y_2$ be the identity map with $X_2 = Y_2 = I$. Then g: $X_1 \times X_2 \to Y_1 \times Y_2$ defined by $g(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is not a closed mapping; because, if $A = \smile\{n \times [0, 1 - 1/(1+|n|)] \mid n = 0, \pm 1, \pm 2, \cdots\}$, we have $(y_0, 1) \in \overline{g(A)} - g(A)$.

3. Let Q be a compact Hausdorff space. We shall say that a topological space X is Q-paracompact, if $X \times Q$ is normal.

Theorem 3.1. *Let Q and Q' be any two compact Hausdorff spaces. If Q' is either a closed subset of Q or a continuous image of Q, then every Q-paracompact space is Q'-paracompact.*

Proof. Suppose that Q' is a continuous image of Q; let f be a continuous mapping of Q onto Q'. Let X be a Q-paracompact space and put $g(x, q) = (x, f(q))$ for $x \in X$, $q \in Q$. Then g is a closed continuous mapping of $X \times Q$ onto $X \times Q'$ by Lemma 2.1. Since X is Q-paracompact, $X \times Q$ is normal, and hence $X \times Q'$ is normal. Therefore X is Q'-paracompact. In case Q' is a closed subset of Q, every Q-paracompact space is clearly Q'-paracompact.

Now we are in a position to prove Theorem 1.2. To prove Theorem 1.2 it is sufficient to prove the following theorem in view of Theorem 1.1.

Theorem 3.2. *Let $\mathfrak{m}$ be an infinite cardinal number. Let X be a topological space. Then the following statements are equivalent.*

(a) *X is $I^{\mathfrak{m}}$-paracompact.*

(b) *X is $C^{\mathfrak{m}}$-paracompact.*

(c) *X is $D^{\mathfrak{m}}$-paracompact.*
Here C is a compact metric space containing at least two points and D is a discrete space consisting of two points.

Proof. $C^{\mathfrak{m}}$ is homeomorphic to a closed subspace of $I^{\mathfrak{m}}$. Hence we have the implication (a)$\rightarrow$(b) by Theorem 3.1. Similarly (b)$\rightarrow$(c) is proved since $D^{\mathfrak{m}}$ is a closed subspace of $C^{\mathfrak{m}}$. Since every compact Hausdorff space with an open base of power$\leqq\mathfrak{m}$ is a continuous image of a closed subset of $D^{\mathfrak{m}}$, $I^{\mathfrak{m}}$ is a continuous image of a closed subset of $D^{\mathfrak{m}}$ and hence the implication (c)$\rightarrow$(a) is proved by Theorem 3.1.

References

[1] C. H. Dowker: On countably paracompact spaces, Can. J. Math., **3**, 219–224 (1951).
[2] K. Morita: Paracompactness and product spaces, forthcoming.