50. Remarks on Katětov's Uniformly 0-dimensional Mappings

By Keiô NAGAMI

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It seems to me that the notion of uniformly 0-dimensional mappings introduced by M. Katětov plays an essential rôle in his dimension theory for non-separable metric spaces [3]. Let R and S be metric spaces (with the metric ρ_1 and ρ_2 respectively) and f a continuous mapping of R into S. According to him, f is called $((\rho_1, \rho_2)$-) uniformly 0-dimensional if the following condition is satisfied.

(*) For any $\varepsilon>0$ there exists a $\delta>0$ such that when $M\subset S$ and dia M[1] $<\delta$, $f^{-1}(M)$ can be decomposed into mutually disjoint relatively open (in $f^{-1}(M)$) sets whose diameters are less than ε.

He proved that for any metric space R with $\dim R$[2] $\leq n$[3] there exists a uniformly 0-dimensional continuous mapping of R into the Euclidean n-space E^n. With the aid of this fundamental theorem he proved the decomposition theorem and in consequence the equality $\dim R=\operatorname{Ind} R$[4] for metric space R. Modifying Katětov's definition, we shall give in this note a definition of uniformly 0-dimensional continuous mappings of normal spaces into normal ones. Let R and S be normal spaces and f a uniformly 0-dimensional continuous mapping, in our sense, of R into S. Then it is the main purpose to show that $\dim R\leq\dim S$ and $\operatorname{Ind} R\leq\operatorname{Ind} S$.

Definition. Let R and S be topological spaces. Let $U=\{\mathfrak{U}_\lambda; \lambda\in\Lambda\}$ and $V=\{\mathfrak{V}_\mu; \mu\in M\}$ be respectively collections of open coverings of R and S. Let f be a continuous mapping of R into S. Then we call that f is (U, V)-*uniformly* 0-*dimensional* if the following condition is satisfied:

(**) For any $\lambda\in\Lambda$ there exists a $\mu\in M$ such that for any $V\in\mathfrak{V}_\mu$ there exists a collection $\{H_\alpha; \alpha\in A\}$ of disjoint open sets of R with $\cup\{H_\alpha; \alpha\in A\}=f^{-1}(V)$ which refines $\mathfrak{U}_\lambda$.

Throughout this note the following notations will be used.

U_F = *the collection of all finite open coverings of* R.

U_B = *the collection of all binary open coverings*[5] *of* R.

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1) dia M denotes the diameter of M.

2) $\dim R$ denotes the covering dimension of R.

3) Throughout this note n denotes a non-negative integer.

4) $\operatorname{Ind} R$ denotes the large inductive dimension of R defined inductively as follows. For the empty set ϕ let $\operatorname{Ind}\phi=-1$. Suppose that $\operatorname{Ind} R'\leq n-1$ is defined. Then $\operatorname{Ind} R\leq n$ if for any pair $F\subset G\,(\subset R)$ of a closed set F and an open set G there exists an open set H with $F\subset H\subset G$ such that $\operatorname{Ind}(\bar{H}-H)\leq n-1$.

5) A covering which consists of two elements is called a binary covering.

V_F = *the collection of all finite open coverings of* S.

V_A = *the collection of all open coverings of* S.

Theorem 1. *Let R and S be normal spaces. If there exists a (U_B, V_F)-uniformly 0-dimensional continuous mapping f of R into S, it holds that* $\dim R \leq \dim S$.

Proof. When $\dim S=\infty$, the theorem is clearly true. Let us consider the case $\dim S\leq n$. Let $\mathfrak{U}=\{U_1,\cdots, U_k\}$ be an arbitrary finite open covering of R. Since R is normal, there exists a closed covering $\{F_1,\cdots, F_k\}$ of R such that $F_i\subset U_i$ for $i=1,\cdots,k$. Since, for any i, $\{G_i, R-F_i\}$ is an element of U_B, there exists a finite open covering $\mathfrak{V}_i=\{V_\alpha;\ \alpha\in A_i\}$ of S such that, for any $\alpha\in A_i$, $f^{-1}(V_\alpha)$ is the sum of two disjoint open sets $H(\alpha, 1)$ and $H(\alpha, 2)$ which satisfy the following conditions: i) $H(\alpha, 1)\subset G_i$, ii) $H(\alpha, 2)\frown F_i=\phi$.

Let $\mathfrak{V}=\{V_\alpha;\ \alpha\in A\}$ be a finite open covering of S of order[6] $\leq n+1$ which refines $\mathfrak{V}_i$ for any i. Let φ_i be a refine-mapping[7] of A into A_i, $i=1,\cdots,k$. Let us consider, for every $\alpha\in A$, an open collection $\mathfrak{D}_\alpha=\{D(\varphi_i(\alpha), j)=f^{-1}(V_\alpha)\frown H(\varphi_i(\alpha), j);\ i=1,\cdots, k,\ j=1, 2\}$. Let $\mathfrak{E}_\alpha=\{E_\gamma;\ \gamma\in\Gamma_\alpha\}$ be a collection of all open sets of type $\bigcap_{i=1}^{k} D(\varphi_i(\alpha), j_i)$. Since $\{D(\varphi_i(\alpha), j);\ j=1, 2\}$ covers $f^{-1}(V_\alpha)$ for every i, $\mathfrak{E}_\alpha$ covers $f^{-1}(V_\alpha)$. Let $E_{\gamma_1}=\bigcap_{i=1}^{k} D(\varphi_i(\alpha), j_i)$ and $E_{\gamma_2}=\bigcap_{i=1}^{k} D(\varphi_i(\alpha), t_i)$ be different elements of $\mathfrak{E}_\alpha$. Then there exists i_0 such that $j_{i_0}\neq t_{i_0}$. Hence $E_{\gamma_1}\frown E_{\gamma_2}\subset D(\varphi_{i_0}(\alpha), j_{i_0})\frown D(\varphi_{i_0}(\alpha), t_{i_0})\subset H(\varphi_{i_0}(\alpha), 1)\frown H(\varphi_{i_0}(\alpha), 2)=\phi$. $\mathfrak{E}_\alpha$ is therefore a mutually disjoint open collection. Moreover we can prove that $\mathfrak{E}_\alpha$ refines $\mathfrak{U}$ as follows. Let E_γ be an element of type $\bigcap_{i=1}^{k} D(\varphi_i(\alpha), 2)$ of $\mathfrak{E}_\alpha$. Then $E_\gamma=\bigcap_{i=2}^{k} D(\varphi_i(\alpha), 2)\frown F_s\subset D(\varphi_s(\alpha), 2)\frown F_s\subset H(\varphi_s(\alpha), 2)\frown F_s=\phi$ for $s=1,\cdots,k$. Hence $E_\gamma\frown(\bigcup_{s=1}^{k} F_s)=\phi$. Since $\{F_1,\cdots, F_k\}$ covers R, we get $E_\gamma=\phi$. Let $E_\delta=\bigcap_{i=1}^{k} D(\varphi_i(\alpha), j_i)$ be an element of $\mathfrak{E}_\alpha$ such that for some i, say i_0, $j_{i_0}=1$. Then $E_\delta\subset D(\varphi_{i_0}(\alpha), 1)\subset H(\varphi_{i_0}(\alpha), 1)\subset G_{i_0}$. Therefore $\mathfrak{E}_\alpha$ is a refinement of $\mathfrak{U}$.

Let $\mathfrak{E}=\{E;\ E\in\mathfrak{E}_\alpha,\ \alpha\in A\}$. Then it can easily be seen that $\mathfrak{E}$ is an open covering of R of order $\leq n+1$ which refines $\mathfrak{U}$. Thus we get $\dim R\leq n$ and the theorem is proved.

Since an arbitrary open covering of a paracompact Hausdorff space of covering dimension $\leq n$ can be refined by an open covering of order $\leq n+1$, we get at once the following proposition by the same argument as employed in the above proof.

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6) $\sup\{|A(x)|;\ A(x)=\{\alpha;\ x\in V_\alpha\in\mathfrak{V}\},\ x\in S\}$ is the order of $\mathfrak{V}$, where $|A(x)|$ denotes the number of indices of $A(x)$.

7) A refine-mapping φ_i of A in A_i is one such that $V_\alpha\subset V_{\varphi_i(\alpha)}$ for every $\alpha\in A$.

Theorem 2. *Let R be a normal space and S be a paracompact Hausdorff space. If there exists a* (U_B, V_A)*-uniformly* 0*-dimensional continuous mapping of R into S, it holds that* $\dim R \leq \dim S$.

The following proposition is also essentially proved in the proof of Theorem 1.

Corollary 1. *Let R and S be normal spaces. If f is a* (U_B, V_F)*-uniformly* 0*-dimensional continuous mapping of R into S, it is* (U_F, V_F)*-uniformly* 0*-dimensional.*

Analogously we get the following.

Corollary 2. *Let R be a normal space and S a paracompact Hausdorff space. If f is a* (U_B, V_A)*-uniformly* 0*-dimensional continuous mapping of R into S, it is* (U_F, V_A)*-uniformly* 0*-dimensional.*

Corollary 3. *Let R be a normal space whose uniform structure is unique. If R is embedded into S, then* $\dim R \leq \dim S$.

Proof. Let $\{G_1, G_2\}$ be an arbitrary binary relatively open covering of R. Then by Doss [1] one of $R-G_1=F_1$ and $R-G_2=F_2$ is compact and hence closed in S. Thus $\overline{F}_1 \frown \overline{F}_2=\phi$ and hence $\{S-\overline{F}_1, S-\overline{F}_2\}$ is an open covering of S. It is evident that $(S-\overline{F}_i)\frown R=G_i$, $i=1,2$, we get $\dim R \leq \dim S$ by Theorem 1.

The first of the following two lemmas can be proved by an analogous method to the proof of Nagami [4, Lemma 5] which is nothing but the second lemma.

Lemma 1. *Let R be a non-empty totally normal space.*[8] *Then* $\operatorname{Ind} R \leq n$ *if and only if for every finite open covering* $\mathfrak{U}$ *of R there exists a mutually disjoint finite open collection* $\mathfrak{V}=\{V\}$ *such that* i) $\overline{\mathfrak{V}}=\{\overline{V}\}$ *refines* $\mathfrak{U}$, ii) $R-\smile V=\smile(\overline{V}-V)$, iii) $\operatorname{Ind}(R-\smile V)\leq n-1$.

Lemma 2. *Let R be a non-empty hereditarily paracompact Hausdorff space, i.e. any of whose subspace is paracompact. Then* $\operatorname{Ind} R \leq n$ *if and only if for every open covering* $\mathfrak{U}$ *there exists a locally finite, mutually disjoint, open collection* $\mathfrak{V}=\{V\}$ *such that* i) $\overline{\mathfrak{V}}$[9] *refines* $\mathfrak{U}$, ii) $R-\smile V=\smile(\overline{V}-V)$, iii) $\operatorname{Ind}(R-\smile V)\leq n-1$.

Theorem 3. *Let R be a normal space and S a non-empty totally normal space. If there exists a* (U_B, V_F)*-uniformly* 0*-dimensional continuous mapping of R into S, it holds that* $\operatorname{Ind} R \leq \operatorname{Ind} S$.

Proof. When $\operatorname{Ind} S=\infty$, the theorem is clearly true. Let us consider the case $\operatorname{Ind} S<\infty$. Let (P_n) be the theorem for the case $\operatorname{Ind} S\leq n$. Then (P_0) has already been proved in Theorem 1, since

8) This notion was introduced by C. H. Dowker [2]. A topological space R is called totally normal if i) R is normal, ii) for any open set G of R there exists a sequence of mutually disjoint closed collections $\mathfrak{F}_i$ which is locally finite in G such that $G=\smile\{F; F\in\mathfrak{F}_i, i=1,2,\cdots\}$.

9) Since $\mathfrak{V}$ is locally finite, $\overline{\mathfrak{V}}$ is also locally finite.

$\operatorname{Ind} S \leq 0$ implies $\dim S \leq 0$. Put the induction assumption that (P_{n-1}), $n>0$, is true. To show the validity of (P_n), let $\operatorname{Ind} S \leq n$.

Let $F \subset G$ be an arbitrary pair of a closed set F and an open set G of R. Then there exists a finite open covering $\mathfrak{B}=\{V_\alpha; \alpha \in A\}$ such that, for every $\alpha \in A$, $f^{-1}(V_\alpha)$ is the sum of two disjoint open sets $W_{\alpha 1}$, $W_{\alpha 2}$ with $W_{\alpha 1} \subset G$ and $W_{\alpha 2} \frown F=\phi$. By Lemma 1 $\mathfrak{B}$ can be refined by a mutually disjoint, finite open collection $\mathfrak{B}_1=\{V_\beta; \beta \in B\}$ such that i) $\overline{\mathfrak{B}}_1$ refines $\mathfrak{B}$, ii) $R-\smile V_\beta=\smile(\overline{V}_\beta-V_\beta)$, iii) $\operatorname{Ind}(R-\smile V_\beta) \leq n-1$. Let $\varphi: B \rightarrow A$ be a refine-mapping. Put $f^{-1}(\overline{V}_\beta-V_\beta) \frown W_{\varphi(\beta),i}=F_{\beta i}$ and $f^{-1}(V_\beta) \frown W_{\varphi(\beta),i}=G_{\beta i}$, $i=1,2$. Let $W_i=\smile\{W_{\alpha i}; \alpha \in A\}$, $i=1$, 2; then $W_1 \subset G$ and $F \frown W_2=\phi$. Let $F_i=\smile\{F_{\beta i}; \beta \in B\}$ and $G_i=\smile\{G_{\beta i}; \beta \in B\}$, $i=1,2$; then $W_i \supset F_i \smile G_i$, $i=1,2$. Let $F_i \smile G_i=H_i$, $i=1,2$; then $\{H_1, H_2\}$ is a closed covering of R. Let D be the open kernel of H_1; then $\overline{D}-D \subset F_1$ and $F \subset D \subset G$. Let $f_1=f \mid \overline{D}-D$. Let U_B' and V_F' be respectively the restrictions of U_B and V_F on $\overline{D}-D$ and $H=R-\smile V_\beta$. Since $\overline{D}-D$ and H are closed, every binary relatively open covering of $\overline{D}-D$ and every finite relatively open covering of H are respectively elements of U_B' and V_F'. Moreover $f_1; \overline{D}-D \rightarrow H$ is evidently (U_B', V_F')-uniformly 0-dimensional, we have $\operatorname{Ind}(\overline{D}-D) \leq n-1$ by the induction assumption. Thus $\operatorname{Ind} R \leq n$ and the theorem is proved.

By Lemma 2 we get the following by an analogous argument to the above.

Theorem 4. *Let R be a normal space and S a non-empty hereditarily paracompact Hausdorff space. If there exists a (U_B, V_A)-uniformly 0-dimensional continuous mapping of R into S, it holds that* $\operatorname{Ind} R \leq \operatorname{Ind} S$.

At the end of this note let us consider the relation between Katětov's original definition of uniformly 0-dimensional mappings and ours. A) Let R be a metrizable space and $U_C=\{\mathfrak{U}_i; i=1,2,\cdots\}$ be a collection of open coverings $\mathfrak{U}_i$ such that, for every $x \in R$, $\{\mathrm{S}(x, \mathfrak{U}_i)$[10]; $i=1,2,\cdots\}$ forms a complete system of neighborhoods of x. Let S be a metric space with the metric ρ_2 and f be a (U_C, V_A)-uniformly 0-dimensional continuous mapping of R into S. Then we can prove $\dim R \leq \dim S$ as follows. When $\dim S=\infty$, the proposition is evidently true. Hence we consider the case $\dim S \leq n$. Let $\mathfrak{B}_i=\{V_{\lambda_i}; \lambda_i \in \Lambda_i\}$, $i=1,2,\cdots$, be a sequence of locally finite open coverings of S of order $\leq n+1$ such that, i) $\overline{\mathfrak{B}}_{i+1}$ refines $\mathfrak{B}_i$, $i=1,2,\cdots$, and ii) for any $\lambda_i \in \Lambda_i$, $f^{-1}(V_{\lambda_i})$ can be decomposed into a mutually disjoint open collection which refines $\mathfrak{U}_i$. Let $\mathfrak{W}_{\lambda_i}=\{W(\lambda_i, \mu); \mu \in M_{\lambda_i}\}$, $\lambda_i \in \Lambda_i$, be a

10) $\mathrm{S}(x, \mathfrak{U}_i)=\smile\{U; x \in U \in \mathfrak{U}_i\}$.

mutually disjoint open collection of R with $f^{-1}(V_{\lambda_i})=\cup\{W(\lambda_i, \mu); \mu\in M_{\lambda_i}\}$ which refines $\mathfrak{U}_i$. Let $\varphi_{i+1,i}$: $\Lambda_{i+1}\to\Lambda_i$ be a refine-mapping, $i=1, 2,\cdots$, and put $\varphi_{j1}=\varphi_{21}\cdots\varphi_{j-1,j-2}\varphi_{j,j-1}$, $j>i$. φ_{jj}: $\Lambda_j\to\Lambda_j$ denotes the identity mapping, $j=1, 2,\cdots$. Then it can easily be seen that $\mathfrak{D}_j=\{W(\lambda_j, \mu_j)\frown W(\varphi_{j,j-1}(\lambda_j), \mu_{j-1})\frown\cdots\frown W(\varphi_{j1}(\lambda_j), \mu_1)$; $\mu_k\in M_{\varphi_{jk}(\lambda_j)}$, $k=1,\cdots, j$, $\lambda_j\in\Lambda_j\}$, $j=1, 2,\cdots$, is a sequence of open coverings of R of order $\leq n-1$ such that, for every $x\in R$, $\{S(x, \mathfrak{D}_j); j=1, 2,\cdots\}$ forms a complete system of neighborhoods of x. Moreover we can prove that $\mathfrak{D}_{j+1}$ is a cushioned-refinement[11] of $\mathfrak{D}_j$, $j=1, 2,\cdots$. Therefore we get $\dim R\leq n$ by Nagami [5, Theorem 2.1]. B) We can construct a metric ρ_1 of R which agrees with the preasigned topology of R such that $\mathfrak{S}_i=\{S_{1/i}(x)=\{y; \rho_1(x, y)<1/i\}; x\in R\}$ refines $\mathfrak{U}_i$, $i=1, 2,\cdots$. If g is a (ρ_1, ρ_2)-uniformly 0-dimensional continuous mapping of R into S, then g is clearly (U_C, V_A)-uniformly 0-dimensional.

References

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11) Let $\mathfrak{E}_1=\{E_\alpha; \alpha\in A\}$ and $\mathfrak{E}_2=\{E_\beta; \beta\in B\}$ be collections of subsets of R. Then $\mathfrak{E}_1$ is called a cushioned-refinement of $\mathfrak{E}_2$, if there exists a refine-mapping $\varphi: A\to B$ such that, for any subset C of A, the closure of $\cup\{E_\alpha; \alpha\in C\}$ is contained in $\cup\{E_\beta; \beta\in\varphi(C)\}$.