66. On Some Properties of Fractional Powers of Linear Operators

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A class of linear operators in a Banach space X is considered in a note by T. Kato.[1] A linear operator A in X is said to be of type (ω, M), if A is densely defined and closed, the resolvent set of $-A$ contains the open sector $|\arg\lambda|<\pi-\omega$, $0<\omega<\pi$, and $\lambda(\lambda+A)^{-1}$ is uniformly bounded in each smaller sector $|\arg\lambda|<\pi-\omega-\varepsilon$, $\varepsilon>0$, in particular $\lambda\|(\lambda+A)^{-1}\|\leq M$, $\lambda>0$. The fractional power A^{α}, $0<\alpha<1$, of A is defined by Kato through

$$(\lambda+A^{\alpha})^{-1}=\frac{\sin\pi\alpha}{\pi}\int_0^{\infty}\frac{\mu^{\alpha}}{\lambda^2+2\lambda\mu^{\alpha}\cos\pi\alpha+\mu^{2\alpha}}(\mu+A)^{-1}d\mu,$$

where λ is in the sector $|\arg\lambda|<(1-\alpha)\pi$, and is shown to be of type $(\alpha\omega, M)$.

K. Yosida[2] gave an example showing that $(A^2)^{1/2}\neq A$ where $-A$ and $-A^2$ are infinitesimal generators of strongly continuous semi-groups. In this paper we shall prove, however, that $(A^{\alpha})^{\beta}=A^{\alpha\beta}$, $0<\alpha$, $\beta<1$. We shall also prove that the semi-group $\{\exp(-tA^{\alpha})\}$ generated by $-A^{\alpha}$ is continuous with respect to α in the uniform operator topology. This result overlaps with A. V. Balakrishnan's result[3] which says that $A^{\alpha}x$ is, for $x\in\mathfrak{D}(A)$, left-continuous at $\alpha=1$.

Theorem 1. Let A be of type (ω, M), then

$$(A^{\alpha})^{\beta}=A^{\alpha\beta},\qquad 0<\alpha,\ \beta<1.$$

Proof. For any μ in the sector $|\arg\mu|<(1-\beta)\pi$

$$(1)\qquad \begin{aligned}(\mu+(A^{\alpha})^{\beta})^{-1}=\frac{1}{(2\pi i)^2}\int_0^{\infty}\left(\frac{1}{\mu+\lambda^{\beta}e^{-i\pi\beta}}-\frac{1}{\mu+\lambda^{\beta}e^{i\pi\beta}}\right)d\lambda\\ \int_0^{\infty}\left(\frac{1}{\lambda+\zeta^{\alpha}e^{-i\pi\alpha}}-\frac{1}{\lambda+\zeta^{\alpha}e^{i\pi\alpha}}\right)(\zeta+A)^{-1}d\zeta.\end{aligned}$$

The double integral being absolutely convergent, we may interchange the order of the integration. Since we obtain

$$\frac{1}{2\pi i}\int_0^{\infty}\left(\frac{1}{\mu+\lambda^{\beta}e^{-i\pi\beta}}-\frac{1}{\mu+\lambda^{\beta}e^{i\pi\beta}}\right)\left(\frac{1}{\lambda+\zeta^{\alpha}e^{-i\pi\alpha}}-\frac{1}{\lambda+\zeta^{\alpha}e^{i\pi\alpha}}\right)d\lambda$$

1) T. Kato: Note on fractional powers of linear operators, Proc. Japan Acad., 36, 94-96 (1960).

2) K. Yosida: Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them, Proc. Japan Acad., 36, 86-89 (1960).

3) A. V. Balakrishnan: Fractional powers of closed operators and the semi-groups generated by them, Pacific J. Math., 10, 419-437 (1960).

$$=\frac{1}{2\pi i}\int_C \frac{1}{\mu+z^\beta}\left(\frac{1}{z-\zeta^\alpha e^{-i\pi\alpha}}-\frac{1}{z-\zeta^\alpha e^{i\pi\alpha}}\right)dz$$

$$=\frac{1}{\mu+\zeta^{\alpha\beta}e^{-i\pi\alpha\beta}}-\frac{1}{\mu+\zeta^{\alpha\beta}e^{i\pi\alpha\beta}},$$

where the path C runs from $\infty e^{i\pi}$ to 0 and from 0 to $\infty e^{-i\pi}$, it follows from (1) that

$$(\mu+(A^\alpha)^\beta)^{-1}=(\mu+A^{\alpha\beta})^{-1}.$$

This shows that $(A^\alpha)^\beta=A^{\alpha\beta}$.

Lemma 1. For each $\varepsilon>0$,

$$\sup\left[\|\lambda(\lambda+A^\alpha)^{-1}\|;\ |\arg\lambda|\leq\pi-\omega-\varepsilon,\ 0<\alpha\leq 1\right]<\infty.$$

Proof. Since A is of type (ω, M), there exists an $M_0<0$ such that

$$\|\lambda(\lambda+A)^{-1}\|\leq M_0$$

for any λ in the sector $|\arg\lambda|\leq\pi-\omega-\varepsilon$. For each α with $0<\alpha<1$ and for each λ in the sector, we have

$$\lambda(\lambda+A^\alpha)^{-1}=\frac{\sin\pi\alpha}{\pi}\int_0^{\infty e^{i\theta}}\frac{\lambda\mu^\alpha}{(\lambda+\mu^\alpha e^{-i\pi\alpha})(\lambda+\mu^\alpha e^{i\pi\alpha})}(\mu+A)^{-1}d\mu$$

where $\theta=\arg\lambda$. Introducing a new integration variable $\xi=\frac{|\mu|^\alpha}{|\lambda|}$, we obtain for $\theta\geq 0$

$$\|\lambda(\lambda+A^\alpha)^{-1}\|\leq\frac{M_0\sin\pi\alpha}{\pi}\left|\int_0^{\infty e^{i\theta}}\frac{\lambda\mu^\alpha}{(\lambda+\mu^\alpha e^{-i\pi\alpha})}d\mu\right|$$

$$=\frac{M_0\sin\pi\alpha}{\pi}\int_0^\infty\frac{d\xi}{1+2\xi\cos(\alpha\theta-\theta-\pi\alpha)+\xi^2}$$

$$=\frac{M_0\sin\pi\alpha\cdot[(\pi-\theta)\alpha+\theta]}{\pi\alpha\sin[(\pi-\theta)(1-\alpha)]},$$

and for $\theta\leq 0$, similarly as above,

$$\|\lambda(\lambda+A)^{-1}\|\leq\frac{M_0\sin\pi\alpha\cdot[(\pi+\theta)\alpha-\theta]}{\pi\alpha\sin[(\pi+\theta)(1-\alpha)]}.$$

Hence the assertion is easily seen.

Lemma 2. For each λ with $|\arg\lambda|<\pi-\omega$

$$\lim_{\alpha\uparrow 1}(\lambda+A^\alpha)^{-1}=(\lambda+A)^{-1}$$

in the sense of the uniform operator topology. The limit holds uniformly in each compact subset of the sector $|\arg\lambda|<\pi-\omega$.

Proof. To prove the assertion it is sufficient to show that

$$\lim_{\alpha\uparrow 1}(1+A^\alpha)^{-1}=(1+A)^{-1}, \tag{2}$$

because of the resolvent equation and Lemma 1.

$$(1+A^\alpha)^{-1}-(1+A)^{-1}$$

$$=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\frac{\mu^{\alpha-1}}{1+2\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}[\mu(\mu+A)^{-1}-(1+A)^{-1}]\,d\mu. \tag{3}$$

For any $\varepsilon>0$ there exists a $\delta>0$ such that

$$\| \mu(\mu+A)^{-1}-(1+A)^{-1} \| \leq \varepsilon$$

if $|\mu-1|\leq\delta$. To see (2), we break up the interval of the integral in (3) into three parts:

$$I_1=\frac{\sin \pi\alpha}{\pi}\int_0^{1-\delta}, \qquad I_2=\frac{\sin \pi\alpha}{\pi}\int_{1-\delta}^{1+\delta}, \qquad I_3=\frac{\sin \pi\alpha}{\pi}\int_{1+\delta}^{\infty}.$$

If we set $\alpha\geq\frac{1}{2}$, then by a simple calculation we obtain

$$\| I_1 \| \leq \frac{4M \sin \pi\alpha \cdot \sqrt{1-\delta}}{\pi(1-\sqrt{1-\delta})^2}, \qquad \| I_3 \| \leq \frac{4M \sin \pi\alpha \cdot \sqrt{1+\delta}}{\pi(\sqrt{1+\delta}-1)^2}.$$

It follows that $\lim_{\alpha\uparrow 1} \| I_1 \| =0=\lim_{\alpha\uparrow 1} \| I_3 \|$. On the other hand it is clear that $\| I_2 \| \leq\varepsilon$. Hence (2) is proved.

Remark. For fixed $\lambda(=e^{\rho+i\theta})$ in the sector $|\arg \lambda|<\pi-\omega$, $(\lambda+A^\alpha)^{-1}$ with $0<\alpha<1$ is defined through

$$(4) \qquad (\lambda+A^\alpha)^{-1}=\frac{\sin \pi\alpha}{\pi}\int_0^{\infty e^{i\theta}} \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha \cos \pi\alpha+\mu^{2\alpha}}(\mu+A)^{-1}d\mu.$$

We shall define $(\lambda+A^\alpha)^{-1}$ through (4) for α in the region

$$\left\{\alpha : \left|\alpha-\frac{1}{2}\left(1+\frac{i\rho}{\pi-\theta}\right)\right|<\frac{\sqrt{(\pi-\theta)^2+\rho^2}}{2(\pi-\theta)} \right.$$

$$\left. \text{and } \left|\alpha-\frac{1}{2}\left(1-\frac{i\rho}{\pi+\theta}\right)\right|<\frac{\sqrt{(\pi+\theta)^2+\rho^2}}{2(\pi+\theta)}\right\}$$

in the complex α-plane, then it is clear that $(\lambda+A^\alpha)^{-1}$ is analytic in the region.

Theorem 2. The semi-group $\exp(-tA^\alpha)$ generated by $-A^\alpha$ is continuous in the uniform operator topology with respect to α in $0<\alpha\leq1$ if $0<\omega<\frac{\pi}{2}$, or in $0<\alpha<\frac{\pi}{2\omega}$ if $\frac{\pi}{2}<\omega<\pi$.

Proof. Suppose that $0<\omega<\frac{\pi}{2}$. Then for any α, $0<\alpha\leq1$, and for any fixed $t>0$

$$(5) \qquad \exp(-tA^\alpha)=\frac{1}{2\pi i}\int_L \frac{e^\zeta}{\zeta}\frac{\zeta}{t}\left(\frac{\zeta}{t}+A^\alpha\right)^{-1}d\zeta$$

where the integration path L runs in the sector $|\arg\zeta|<\pi-\omega$ from $\infty e^{-i\theta_1}$ to $\infty e^{-i\theta_2}$ with $\frac{\pi}{2}<\theta_1$, $\theta_2<\pi-\omega$. The uniform boundedness of $\frac{\zeta}{t}\left(\frac{\zeta}{t}+A^\alpha\right)^{-1}$, the continuity of $\left(\frac{\zeta}{t}+A^\alpha\right)^{-1}$ for α and the integrability of $\left|\frac{e^\zeta}{\zeta}\right|$ over L show that $\exp(-tA^\alpha)$ is continuous in α.

In the case of $\frac{\pi}{2}\leq\omega<\pi$, we can prove the assertion as above by taking integration path L in (5) appropriately.