11. *On the Weak Definability in Set Theory*

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We refer to [4] and [5] as to the notions and notations throughout this paper. Here we restate some of them. A "set theory" means a set theory in the first order predicate calculus, containing only the predicate $\in$, logical symbols, bound variables and finitely or infinitely many individual constants. If T is a set theory in this sense containing $a_0, a_1, \cdots$ as individual constants, we call T a set theory with $a_0, a_1, \cdots$. Let T be a set theory with $a_0, a_1, \cdots$ and B_T be the class consisting of all $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$, where $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$ contains only logical symbols, the predicate $\in$, bound variables and $a_0, a_1, \cdots$. $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$ will be abbreviated as $\{x\}\mathfrak{A}(x)$ if no confusion is to be feared. T is called 'definite', if it satisfies the following conditions: 1) T is complete. (I.e. for any closed formula $\mathfrak{A}$ (which, more precisely, should be written as $\mathfrak{A}(a_0, a_1, \cdots)$) either $\mathfrak{A}$ or $\neg\mathfrak{A}$ belongs to T.) 2) If $\exists x\mathfrak{A}(x)$ belongs to T, then there exists a formula $\exists x\mathfrak{B}(x)$ such that $\forall x\forall y(\mathfrak{B}(x)\wedge\mathfrak{B}(y)\vdash x=y)$ and $\exists x(\mathfrak{A}(x)\wedge\mathfrak{B}(x))$ belong to T.

Let $\{x\}\mathfrak{A}(x)$ and $\{x\}\mathfrak{B}(x)$ belong to B_T. We say '$\{x\}\mathfrak{B}(x)$ belongs to the same class with $\{x\}\mathfrak{A}(x)$ relative to T', if and only if $\forall x(\mathfrak{A}(x)\dashv\vdash\mathfrak{B}(x))$ belongs to T. The class which contains $\{x\}\mathfrak{A}(x)$ is written $(\{x\}\mathfrak{A}(x))$ and $\{x\}\mathfrak{A}(x)$ is said to represent the class. A class $(\{x\}\mathfrak{A}(x))$ is said to be definite with respect to T, if $\exists x\mathfrak{A}(x)$ and $\forall x\forall y(\mathfrak{A}(x)\wedge\mathfrak{A}(y)\vdash x=y)$ belong to T. $A(T)$ is defined to be the set of all the definite classes. Let $(\{x\}\mathfrak{A}(x))$ and $(\{x\}\mathfrak{B}(x))$ be two elements of $A(T)$. Then

$$(\{x\}\mathfrak{A}(x)) \in^*_T (\{x\}\mathfrak{B}(x))$$

is defined to mean that '$\exists x\exists y(\mathfrak{A}(x)\wedge\mathfrak{B}(y)\wedge x\in y)$ belongs to T'.

In [5] we considered a set theory $T_C(a)$. It contains all the elements of a set a in C (='Cantor's Absolute') as individual constants, and consists of all the formulas which are true in C. Here we consider the set theory $T_C(On)$ which contains all ordinal numbers as individual constants.

A set in C is called *weakly definable* (or a weakly definable set) if it is definable in $T_C(On)$.

We present the following two statements as the first and the second weak definability principles:

1. Every set is weakly definable.
2. Every non-empty weakly definable set contains a weakly

definable set as an element.

It is clearly seen that, if $V=L$ of Gödel [1] holds, then both of the weak definability principles also hold. Further, it is clear that the first weak definability principle implies the second one. In this paper we shall show that the second weak definability principle implies the first one and the axiom of choice (in the strong form).

In the following we shall assume the second weak definability principle.

Proposition 1. $T_C(On)$ is definite.

Proof. $T_C(On)$ is clearly complete. Suppose $\exists x \mathfrak{A}(x, \alpha)$ belongs to $T_C(On)$. Then a set x consisting of all z's such that $\mathfrak{A}(z, \alpha) \wedge \forall y(\mathfrak{A}(y, \alpha) \vdash r(y) \geqq r(z))$ (where $r(a)$ is the rank of a) is weakly definable. By the second weak definability principle, there exists a weakly definable set z belonging to x. Let $\{x\}\mathfrak{B}(x, \beta)$ represent the definite class which defines z. Then $\mathfrak{A}(z, \alpha) \wedge \mathfrak{B}(z, \beta)$ holds. Thus we see that $\forall y \forall z(\mathfrak{B}(y, \beta) \wedge \mathfrak{B}(z, \beta) \vdash y=z)$ and $\exists z(\mathfrak{A}(z, \beta) \wedge \mathfrak{B}(z, \beta))$ belong to $T_C(On)$.

By this and the proposition proved in [5], we have

Proposition 2. Let $a_1, \cdots, a_n$ be elements of $A(T_C(On))$ represented by $\{x\}\mathfrak{A}_1(x, \alpha_1), \cdots, \{x\}\mathfrak{A}_n(x, \alpha_n)$ respectively. Then $\mathfrak{A}(a_1, \cdots, a_n)$ is satisfied in $\langle A(T_C(On)), \in^*_{T_C(On)}\rangle$ if and only if

$$\exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1, \alpha_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n, \alpha_n) \wedge \mathfrak{A}(x_1, \cdots, x_n))$$

belongs to $T_C(On)$.

Let C_0 be the class of all the weakly definable sets.

Proposition 3. For any elements $c_1, \cdots, c_n$ of C_0, $\mathfrak{A}(c_1, \cdots, c_n)$ holds in C_0 if and only if it holds in C. (In other word, C is an arithmetical extension of C_0 (cf. [7]).)

Proof. It is easily seen that $\langle C_0, \in\rangle$ (where $\in$ means $\in_{C_0}$) is isomorphic to $\langle A(T_C(On)), \in^*_{T_C(On)}\rangle$. For simplicity we shall identify the corresponding elements. Let $c_1, \cdots, c_n$ be elements of C_0. Then

$\mathfrak{A}(c_1, \cdots, c_n)$ is satisfied in $\langle C_0, \in\rangle$.

$\rightleftarrows \mathfrak{A}(c_1, \cdots, c_n)$ is satisfied in $\langle A(T_C(On)), \in^*_{T_C(On)}\rangle$.

$\rightleftarrows \exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1, \alpha_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n, \alpha_n) \wedge \mathfrak{A}(x_1, \cdots, x_n))$,

where $\{x\}\mathfrak{A}_i(x, \alpha_i)$ represents $c_i (1 \leqq i \leqq n)$, belongs to $T_C(On)$ (by Proposition 2).

$\rightleftarrows \exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1, \alpha_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n, \alpha_n) \wedge \mathfrak{A}(x_1, \cdots, x_n))$

is satisfied in $\langle C, \in\rangle$.

$\rightleftarrows \mathfrak{A}(c_1, \cdots, c_n)$ is satisfied in $\langle C, \in\rangle$.

In a well-known way we can define a formula $\mathfrak{D}(a, n, x, \beta)$ with the following properties:

(1) $\mathfrak{D}(a, n, x, \beta)$ is constructed by the predicate $\in$, logical symbols, bound variables and free variables a, n, x and β only.

(2) 'For any a, x, β and any formula $\mathfrak{A}(x, \beta)$ with only free variables x and β,

$$\mathfrak{D}(a, \ulcorner\mathfrak{A}\urcorner, x, \beta) \rightleftarrows x \in a \wedge \beta \in a \wedge \mathfrak{A}^a(x, \beta),$$

where $\ulcorner\mathfrak{A}\urcorner$ stands for the Gödel number of the formula $\mathfrak{A}$ (cf. [**2**]). Moreover we can define $\tilde{\mathfrak{D}}(j, x)$ such that

$$\tilde{\mathfrak{D}}(j(n, j(\alpha, \beta)), x) \rightleftarrows \mathfrak{D}(R(\alpha), n, x, \beta) \quad (n = \ulcorner\mathfrak{A}\urcorner),$$

where j is an isomorphism of On^2 to On (cf. P in [**1**, p. 29] or j in [**6**]).

Let x be an element of C_0. Then there exists a definite $\{y\}\mathfrak{A}(y, \beta)$ such that $\mathfrak{A}(x, \beta)$. By means of [**3**], there exists α such that x and β belong to the set $R(\alpha)$ of all x with $r(x) < \alpha$, and $\langle R(\alpha), \in_{R(\alpha)}\rangle$ is a model of $T_C(R(\alpha))$. Then $\langle C, \in\rangle$ can be considered as an arithmetical extension of $\langle R(\alpha), \in_{R(\alpha)}\rangle$. Let n be $\ulcorner\mathfrak{A}\urcorner$ and α_0 be $j(n, j(\alpha, \beta))$. Then we have

$$\tilde{\mathfrak{D}}(\alpha_0, x) \wedge \forall y(\tilde{\mathfrak{D}}(\alpha_0, y) \vdash x = y).$$

Thus we see that, for any $x \in C_0$,

$$\exists \alpha(\tilde{\mathfrak{D}}(\alpha, x) \wedge \forall y(\tilde{\mathfrak{D}}(\alpha, y) \vdash x = y))$$

is true (i.e. is satisfied in $\langle C, \in\rangle$). By Proposition 3 this is also true in C_0. Since above x is an arbitrary element of C_0.

$$\forall x \exists \alpha(\tilde{\mathfrak{D}}(\alpha, x) \wedge \forall y(\tilde{\mathfrak{D}}(\alpha, y) \vdash x = y))$$

is true in C_0. Using Proposition 3 again we see that

$$\forall x \exists \alpha(\tilde{\mathfrak{D}}(\alpha, x) \wedge \forall y(\tilde{\mathfrak{D}}(\alpha, y) \vdash x = y))$$

is also true in C. This implies that every set is weakly definable. Let x correspond to the least α such that

$$\tilde{\mathfrak{D}}(\alpha, x) \wedge \forall y(\tilde{\mathfrak{D}}(\alpha, y) \vdash x = y).$$

Then we see that the axiom of choice holds.

Remark. It is clearly seen that, if $V \neq L$, then there exists a weakly definable (even definable) set which is not contained in L. (For the proof, consider the set

$$\{x \mid x \notin L \wedge \forall y(r(y) < r(x) \vdash y \in L)\}.)$$

Here the author wishes to give some remarks on [4] and [**5**].

1. The conclusion of [**5**] is merely a special case of Theorem 5.2 of Montague-Vaught's paper [**3**].

2. Dana Scott pointed out that 'T_C is not maximal in the sense of [4]' is simply proved without using $V = L$ as follows:

Suppose T_C be maximal. A formula $\mathfrak{A}(a)$ is defined to be 'a is a super-complete model of Zermelo-Fraenkel's set theory and the theory, which is satisfied in $\langle a, \in_a\rangle$, is maximal'. Let a_0 be the minimam set satisfying $\mathfrak{A}(a_0)$ and T_0 be the complete theory which is satisfied in $\langle a_0, \in_{a_0}\rangle$. Then a_0 is definable in T_C, hence $T_0 \prec T_C$, which is a contradiction.

We had better say that the main result of [4] is this:

The complete set theories with $V=L$ and a regular model are well ordered by the embedding relation $\prec$. Note in passing that 'the degree of unsolvability of T_1' $\leqq$ 'the degree of unsolvability of T_2', if $T_1 \prec T_2$.

3. By the similar argument as in 2, we have the following proposition.

There exists a theory T such that $T \prec T_C$ and

$\forall x \exists y (x \in y \wedge$ 'y is a super-complete model of T').

4. Also we have the following proposition.

There exists an ordinal α_0 such that $R(\alpha_0)$ is a natural model of T_C and C is not an arithmetical extension of $R(\alpha_0)$.

Proof. If there does not exist any such ordinal, then in virtue of Sandwich Theorem of [2] T_C is characterized by the following conditions on T:

If $\alpha < \beta$ and $R(\alpha)$ and $R(\beta)$ are natural models of T, then $R(\beta)$ is an elementary extension of $R(\alpha)$.

$\forall x \exists \alpha (x \in R(\alpha) \wedge$ '$R(\alpha)$ is an natural model of T').

Hence follows a contradiction by Theorem 1, II of [8].

References

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