61. On Circle and Quasi-Hausdorff Methods of Summability of Fourier Series

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§1. Euler and circle methods of summability of Fourier series. Here the author wishes to discuss the circle method of summability and other quasi-Hausdorff methods of summability of Fourier series. At the beginning we remember the Euler method of summability. It associates with a given sequence $\{s_n\}$ the means

$$\sigma_{n,r}=\sigma_n=\sum_{\nu=0}^{n}\binom{n}{\nu}r^{\nu}(1-r)^{n-\nu}s_\nu,\quad n=0,1,2,\cdots,$$

where r is a constant which satisfies $0<r\leq 1$. We denote this method as (ε, r). The case $r=1$ corresponds to ordinary convergence. The Lebesgue constants for this method of Fourier series are given by L. Lorch [1] and A. E. Livingston [2].

Theorem 1. *The Lebesgue constants for the* (ε, r) *method are given by*

$$L(n;r)=\frac{2}{\pi^2}\log\left|\frac{2nr}{1-r}\right|+A+o(1),\ \textit{as}\ n\to\infty,\ \textit{where}$$

$$A=-\frac{2C}{\pi^2}+\frac{2}{\pi}\int_0^1\frac{\sin u}{u}\,du-\frac{2}{\pi}\int_1^\infty\left\{\frac{2}{\pi}-|\sin u|\right\}\frac{du}{u}.$$

C is the Euler-Mascheroni constant.

The Gibbs phenomenon of the Fourier series $\sum_{n=1}^{\infty}\frac{\sin nt}{n}$ for this method are investigated by O. Szász [3].

Theorem 2. *If we put* $s_0=0$, $s_n=\sum_{\nu=1}^{n}\frac{\sin \nu t}{\nu}$, *then we have*

$$\lim_{n\to\infty}\sigma_n(t_n)=\int_0^{\tau r}\frac{\sin y}{y}\,dy,\ \textit{as}\ nt_n\to\tau\ \textit{and}\ nt_n^2\to 0.$$

On the other hand the circle method of summability associates with a given sequence $\{s_n\}$ the means

$$\sigma^*_{n,r}=\sigma^*_n=\sum_{\nu=n}^{\infty}\binom{\nu}{n}r^{n+1}(1-r)^{\nu-n}s_\nu,\quad n=0,1,2,\cdots,$$

where r is a constant which satisfies $0<r\leq 1$. The case $r=1$ corresponds to ordinary convergence. We denote this method as (γ, r). The Lebesgue constants for this method of Fourier series are given by the author [4].

Theorem 3. *The Lebesgue constants for the* (γ, r) *method are given by*

$$L^*(n;r)=\frac{2}{\pi^2}\log\left|\frac{2n}{1-r}\right|+A+o(1),\ as\ n\to\infty,$$

where A is the same constant as before.

The Gibbs phenomenon of the Fourier series $\sum_{n=1}^{\infty}\frac{\sin nt}{n}$ for this method are given by the author [5].

Theorem 4. *If we put $\{s_n\}$ as in Theorem 2, then we have*

$$\lim_{n\to\infty}\sigma_n^*(t_n)=\int_0^{\tau/r}\frac{\sin y}{y}dy,\ as\ nt_n\to\tau\ and\ nt_n^2\to 0.$$

Here we see the relation between the (ε, r) method and the (γ, r) method. If we replace r in the Lebesgue constants and the Gibbs phenomenon for the (ε, r) method by $\frac{1}{r}$, then we get the same constants and the same phenomenon for the (γ, r) method, i.e. $L\left(n;\frac{1}{r}\right)=L^*(n;r)+o(1)$ and so on.

We can see that the transformation matrix of the (ε, r) method is given by

$$E_r=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-r & r & 0 & 0 & \cdots \\ (1-r)^2 & \binom{2}{1}r(1-r) & r^2 & 0 & \cdots \\ (1-r)^3 & \binom{3}{1}r(1-r)^2 & \binom{3}{2}r^2(1-r) & r^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Similarly the transformation matrix of the (γ, r) method is given by

$$\Gamma_r=\begin{pmatrix} r & r(1-r) & r(1-r)^2 & r(1-r)^3 & \cdots \\ 0 & r^2 & \binom{2}{1}r^2(1-r) & \binom{3}{1}r^2(1-r)^2 & \cdots \\ 0 & 0 & r^3 & \binom{3}{2}r^3(1-r) & \cdots \\ 0 & 0 & 0 & r^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Between the transformation matrix E_r of the (ε, r) method and the transformation matrix Γ_r of the (γ, r) method we see the relation $\Gamma_r=rE_r^*$, where E_r^* is the transposed matrix of E_r. The close relations between the (ε, r) method and the (γ, r) method are investigated by W. Meyer-König [6], J. Taghem [7], P. Vermes [8] and so on.

§2. **Hausdorff and quasi-Hausdorff methods of summability of Fourier series.** We can understand the Euler method as a special case of the Hausdorff method of summability. We define the Hausdorff means of a given sequence $\{s_n\}$ by

$$h_n=\sum_{\nu=0}^{n}\binom{n}{\nu}s_\nu\int_0^1 r^\nu(1-r)^{n-\nu}\,d\psi(r),\ n=0,1,2,\cdots,$$

where $\psi(r)$ is of bounded variation in $0\leq r\leq 1$. This transformation is regular if, and only if,

$$\int_0^1 d\psi(r)=\psi(1)-\psi(0)=1$$

and $\psi(r)$ is continuous at $r=0$, [9]. If we choose

$$\psi(u)=\begin{cases}0 \text{ for } 0\leq u<r\\ 1 \text{ for } r\leq u\leq 1,\end{cases}$$

then the Hausdorff means reduce to the Euler means. For the Hausdorff means we know the Szász theorem [10].

Theorem 5. *If we put* $s_0=0$ *and* $s_n=\sum_{\nu=1}^{n}\frac{\sin \nu t}{\nu}$, *then*

$$\lim_{n\to\infty} h_n(t_n)=\int_0^1\int_0^\tau \frac{\sin ry}{y}\,dy\,d\psi(r),\ \textit{as}\ nt_n\to\tau\ (\leq\infty).$$

On the other hand L. Lorch and D. J. Newman [11] proved

Theorem 6. *Let* $L_h(n;\psi)$ *denote the nth Lebesgue constant for the regular Haudorff method with weight function* $\psi(t)$. *Then*

$$L_h(n;\psi)=C(\psi)\log n+O(\log n),\ \textit{as}\ n\to\infty,\ \textit{where}$$
$$C(\psi)=(2/\pi^2)\,|\,\psi(1)-\psi(1-)\,|+$$
$$+(1/\pi)\mathcal{M}\{|\sum_k[\psi(\xi_k+)-\psi(\xi_k-)]\sin\xi_k x\,|\}.$$

Here ξ_k *is the kth discontinuity (jump) of* $\psi(t)$ *and the summation extends over all such (possibly countably infinite) values;* $\mathcal{M}\{f(x)\}$ *represents, as usual, the mean value of the almost periodic function* $f(x)$. *Furthermore the error term* $o(\log n)$ *is "best possible" and cannot be improved even for the case of an increasing absolutely continuous* $\psi(t)$.

As is well known G. H. Hardy summarized the Hausdorff method of summability in his book "Divergent series". M. S. Ramanujan studied the quasi-Hausdorff method of summability in complete detail similarly as Hardy did. (See [12, 13, 14].) Consequently we can deal with these two methods as a pair. We define the quasi-Hausdorff means of a given sequence $\{s_n\}$ by

$$h_n^*=\sum_{\nu=n}^{\infty}\binom{\nu}{n}s_\nu\int_0^1 r^{n+1}(1-r)^{\nu-n}\,d\psi(r),\ n=0,1,2,\cdots,$$

where $\psi(r)$ is of bounded variation in $0\leq r\leq 1$. This transformation is regular if and only if

$$\int_0^1 d\psi(r)=\psi(1)-\psi(0)=1.\quad \text{(See [12].)}$$

We can understand the circle method as a special case of the quasi-Hausdorff method. As O. Szász generalized Theorem 2 and got Theorem 5, we can expect to generalize Theorem 4 and to get

Theorem 7. *If we put* $s_0=0$ *and* $s_n=\sum_{\nu=1}^{n}\frac{\sin \nu t}{\nu}$, *then we have*

$$\lim_{n\to\infty} h_n^*(t_n)=\int_0^1\int_0^\tau \frac{\sin y/r}{y}dy d\psi(r), \text{ as } nt_n\to\tau.$$

Similarly we can expect to get the theorem about the Lebesgue constants for the quasi-Hausdorff method.

§3. **Problems.** The relations between Hausdorff and quasi-Hausdorff methods of summability for a same weight function $\psi(r)$ are very interesting. We can suppose these two methods as a dual or a conjugate of each other. We shall introduce here several problems.

Problem 1. If we put $\psi(u)=1-(1-u)^p$ in the definition of the Hausdorff method, then we get the Cesàro (C, p) method. For the same weight function $\psi(u)$ in the definition of the quasi-Hausdorff method we get the (C^*, p) method. B. Kuttner gave the relations between (C, p) and (C^*, p) methods. (See [15].)

Theorem 8. *The proposition* (C, p) *implies* (C^*, p) *is false when* $0<p<2$, $p\neq 1$, *but is true when* $p=1$ *or* $p=2$.

Theorem 9. *If* $p>0$ *(p an integer), then* (C^*, p) *implies* (C, p).

We do not know whether or not (C^*, p) implies (C, p) when p is not an integer. Here we denote "method A implies method B" when any sequence summable A is summable B to the same sum. For the (C, p) means we know the Cramér theorem about the Gibbs phenomenon of the Fourier series $\sum_{n=1}^{\infty}\frac{\sin nt}{n}$. (See [16].)

Theorem 10. *There exists a number* p_0, $0<p_0<1$, *with the following property: The* (C, p) *means of* $\sum_{n=1}^{\infty}\frac{\sin nt}{n}$ *present the Gibbs phenomenon at* $t=0$ *for* $p<p_0$, *but not for* $p\geq p_0$.

Is the similar result true or not for the (C^*, p) means?

Problem 2. When series $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ are given, then we denote

$$c_p=\sum_{m+n=p}a_m b_n,\ p=0, 1, 2,\cdots,$$

and we say $\sum_{p=0}^{\infty}c_p$ as the Cauchy product series of $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$. Concerning the Cauchy product series we know the classical Abel, Mertens and Cauchy theorems. For the Euler method of summability of this series K. Knopp proved the theorems of Abel's and Mertens' types, see [17], and H. Hara proved the theorem of Cauchy's type, see [18]. We can prove also the theorems of these three types for the circle method, see [19].

When a series $\sum_{n=0}^{\infty}a_n$ is given, let s_n, $n=0, 1, 2,\cdots$, be the partial sums of this series. If the sequence $\{s_n\}$ is summable to A by the (γ, r) method, then we say $\sum a_n=A(\gamma, r)$. This is the same with

$\sum_{n=0}^{\infty} a_n^* = A$, where

$$a_n^* = \sum_{\nu=n}^{\infty} \binom{\nu}{n} r^n (1-r)^{\nu-n} a_\nu, \quad n=0,1,2,\cdots \text{ . (See [9].)}$$

If $\sum_{n=0}^{\infty} a_n^*$ converges absolutely to A, then we say $\sum a_n = A(|\gamma, r|)$. Then we can prove the following theorems.

Theorem 11. *If* $\sum a_n = A(\gamma, r)$, $\sum b_n = B(\gamma, r)$ *and* $\sum c_n = C(\gamma, r)$, *then* $AB = C$.

Theorem 12. *If* $\sum a_n = A(|\gamma, r|)$ *and* $\sum b_n = B(\gamma, r)$, *then* $\sum c_n = C(\gamma, r)$ *and* $C = AB$.

Theorem 13. *If* $\sum a_n = A(|\gamma, r|)$ *and* $\sum b_n = B(|\gamma, r|)$, *then* $\sum c_n = C(|\gamma, r|)$ *and* $C = AB$.

Here we meet the following problems. For the regular Hausdorff and the regular quasi-Hausdorff means are the theorems of these three types true or not?

Problem 3. It seems to the author that the Hausdorff and the quasi-Hausdorff methods of summability are the most fundamental among the various methods of summability such as the elementary particles in theoretical physics. So we might be able to represent a known method as a combination of these two methods, product and so on. This problem is too vague, but if we can prove this it shall be very interesting.

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