55. A Note on Hasse-Witt Matrices of Algebraic Curves of Positive Characteristic p

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1. Let A be an Abelian variety over a field k of positive characteristic p and $k(A)$ be the field of rational functions on A over k. Let A operate on $k(A)$ as follows:

$$(af)(x)=f(a^{-1}x) \quad (f\in k(A);\, a, x\in A).$$

A derivation D of $k(A)/k$ is called an invariant derivation if $a\circ D\circ a^{-1}=D\,(a\in A)$. It is known that the module of invariant derivations is a k-vector space $\mathfrak{M}(A)$ of the same dimension of A over k and for any $D\in\mathfrak{M}(A)$ D^p is also an invariant derivation. The dual space $(\mathfrak{M})^*(A)$ is the vector space of all invariant 1-differentials of $k(A)/k$. Let $\{D_1,\cdots, D_n\}$ be a k-base of $\mathfrak{M}(A)$ and $\{\omega_1,\cdots,\omega_n\}$ be the dual k-base of $\mathfrak{M}^*(A)$ with respect to $\{D_1,\cdots, D_n\}$, i.e. $\{\omega_1,\cdots,\omega_n\}$ is a k-base of $\mathfrak{M}^*(A)$ such that

$$\omega_i(D_j)=\begin{cases}1, & (i=j)\\ 0, & (i=j).\end{cases}$$

Then $\omega_i(D_j^p)$ $(1\leq i, j\leq n)$ are elements in k. We shall call the square matrix $(\omega_i(D_j^p))$ the Hasse-Witt matrix of A with respect to the base $\{D_1,\cdots, D_n\}$.

In the present note we shall notice that, if J is a Jacobian variety of an algebraic curve Γ, the Hasse-Witt matrix is nothing else than a Hasse-Witt matrix of Γ introduced by Hasse and Witt.[1]

2. We shall first recall the definition of a Hasse-Witt matrix of an algebraic curve. Let Γ be a non-singular complete curve defined over a field k of characteristic $p(>0)$ and g be the genus of Γ. For the sake of simplicity we may assume that k is algebraically closed. Let $P_1+\cdots+P_g$ be a non-special divisor of degree g on Γ and $t_1,\cdots, t_g$ be local parameters at $P_1,\cdots, P_g$, respectively, and $\{\omega_1,\cdots,\omega_g\}$ be a base of k-vector space of all differentials of 1st kind on Γ. Let $\omega_i=(\sum_{\nu} a_{ij}^{(\nu)}t_j^{\nu})dt_j$ be the t_j-expansion of ω_i $(1\leq i, j\leq g)$, and $B^{(\nu)}$ be the matrix of which (i, j)-element is $a_{ij}^{(\nu)}$ $(\nu=0, 1, 2,\cdots)$. Then Hasse and Witt defined the Hasse-Witt matrix H_Γ of Γ by $B^{(0)-1}B^{(p-1)}$.

We shall choose local parameters $t_1,\cdots, t_g$ respectively at $P_1,\cdots, P_g$ such that

1) See [1].

(1) $$t_i \equiv 1 \mod P_j (i \neq j).$$

By Riemann-Roch Theorem such a system of local parameters always exists. We denote by $\Gamma = \Gamma_1, \Gamma_2, \cdots, \Gamma_g$ the copies of Γ and by $k(\Gamma_i)$ the field of rational functions on Γ_i over k, where $k(\Gamma_1), \cdots, k(\Gamma_g)$ are independent over k. We denote by σ_i the canonical isomorphism of Γ onto Γ_i and write simply $t_j^{(i)}$ instead of $\sigma_i(t_j)$. We denote by J the Jacobian variety of Γ and denote by φ the canonical mapping of Γ into J. We denote by the same notation φ the mapping of $\Gamma_1 \times \cdots \times \Gamma_g$ into J defined by $\varphi(\sigma_1(Q) \times \cdots \times \sigma_g(Q_g)) = \sum_{i=1}^{g} \varphi(Q_i)$. Then the field of rational functions $k(\Gamma_1 \times \cdots \times \Gamma_g)$ on $\Gamma_1 \times \cdots \times \Gamma_g$ is a finite separable normal algebraic extension of the field of rational functions $k(J)$ on J whose galois group is induced by the permutations of the factors $\Gamma_1, \cdots, \Gamma_g$. Namely $k(J)$ is the subfield of $k(\Gamma_1 \times \cdots \times \Gamma_g)$ consisting of all f such that $f(\sigma_1(Q_1) \times \cdots \times \sigma_g(Q_g)) = f(\sigma_1(Q_{i_1}), \cdots, \sigma_g(Q_{i_g}))$ for every permutation $\begin{pmatrix} 1, 2, \cdots, g \\ i_1, i_2, \cdots, i_g \end{pmatrix}$. Since every derivation of $k(J)$ is uniquely extended to a derivation of $k(\Gamma_1 \times \cdots \times \Gamma_g)$, we shall use same notations for restricted derivations and extended derivations. Since $\Gamma_1 \times \cdots \times \Gamma_g$ and J are non-singular and φ is regular, the image $\varphi^*(\Omega)$ of an invariant 1-differential ω on J is a 1-differential of 1st kind on $\Gamma_1 \times \cdots \times \Gamma_g$. Such a 1-differential of 1st kind $\Omega' = \varphi^*(\Omega)$ is characteriged by the invariantness for permutations of indices:

$$\Omega'(\sigma_1(Q_1), \cdots, \sigma_g(Q_g)) = \Omega'(\sigma_1(Q_{i_1}), \cdots, \sigma_g(Q_{i_g})).$$

Namely Ω' is the image of an invariant 1-differential on J if and only if $\omega' = \sum_{\nu=1}^{g} \sigma_\nu(\omega)$ with a differential ω of 1st kind on Γ. Moreover φ^* is a monomorphism of the module of invariant differentials on J into the module of differentials on $\Gamma_1 \times \cdots \times \Gamma_g$. In the following we shall identify Ω with $\varphi^*(\Omega)$.

Let $\{\omega_1, \omega_2, \cdots, \omega_g\}$ be a k-base of the module of all the differentials of 1st kind on Γ and put

$$\Omega_i = \sum_{\nu=1}^{g} \sigma_\nu(\omega_i),$$

(2) $$s_j = t_j^{(j)}, \quad (1 \leq i, j \leq g).$$

Since the module of all the invariant 1-differentials is the dual module of all the invariant derivations, there exists a k-base $\{D^{(1)}, \cdots, D^{(g)}\}$ of invariant derivations such that $\Omega_i(D^{(j)}) = 1 \ (i=j),\ 0 \ (i \neq j)$, where $\{D^{(j)}\}$ are considered as extended derivations of $k(\Gamma_1 \times \cdots \times \Gamma_g)$ induced by derivations $\{D^{(i)}\}$ of $k(J)$.

3. We shall show that the k-matrix $(\Omega_i(D^{(j)p}))$ is the Hasse-Witt matrix of the algebraic curve. We denote by

(3) $$\omega_i = \left(\sum_{\nu=0}^{\infty} a_{ij}^{(\nu)} t_j^\nu \right) dt_j$$

the t_j-expansion of ω_i $(1\leq i, j\leq g)$. Then we have

$$(4)\qquad \Omega_i=\Big(\sum_{j=1}^{g} a_{ij}^{(\nu)} t_j^{(j)\nu}\Big)dt_j^{(j)}=\Big(\sum_{j=1}^{g} a_{ij}^{(\nu)} s_j^{\nu}\Big)ds_j,\ (1\leq i\leq g).$$

Since $\{s_1,\cdots,s_g\}$ is a system of uniformizing parameters of $\Gamma_1\times\cdots\times\Gamma_g$ at $P_1\times P_2\times\cdots\times P_g$, there exists a base $\{D_{s_1},\cdots,D_{s_g}\}$ of the space of all the derivations of $k(\Gamma_1\times\cdots\times\Gamma_g)/k$ such that

$$(5)\qquad D_{s_i}(s_j)=\delta_{ij},\ (1\leq i, j\leq g).$$

We put

$$(6)\qquad \alpha_{ij}(s_j)=\sum_{\nu=0}^{\infty} a_{ij}^{(\nu)} s_j^{\nu},\ (1\leq i, j\leq g),$$

and

$$(7)\qquad D^{(j)}=\sum_{l=1}^{g}\beta_{lj}(s)D_{s_l},\ (1\leq j\leq g).$$

Then, since

$$(8)\qquad \Omega_i=\sum_{l=1}^{g}\alpha_{il}(s_l)ds_l,\ (1\leq i\leq g),$$

we have

$$\Omega_i(D^{(j)})=\Big(\sum_{l=1}^{g}\alpha_{il}ds_l\Big)\Big(\sum_{l=1}^{g}\beta_{lj}D_{s_l}\Big)=\sum_{i=1}^{g}\alpha_{il}\beta_{lj}=\begin{cases}1 & (1=j)\\ 0 & (i\neq j)\end{cases}.$$

This shows (β_{ij}) is the inverse matrix of (α_{ij}). Hence we have

$$(9)\qquad D_{s_i}=\sum_{l=1}^{g}\alpha_{il}(s_l)D^{(l)},\ (1\leq i\leq g).$$

We shall first prove

$$(10)\qquad D_{s_j}^{p}=\sum_{l=1}^{g}D_{s_j}^{p-1}(\alpha_{lj})D^{(l)}+\sum_{l=1}^{g}\alpha_{lj}^{p}D^{(l)p},\ (1\leq j\leq g).$$

Since $D_{s_j}^p$ is also a derivations, $D_{s_j}^p=\Big(\sum_{l=1}^{g}\alpha_{jl}(s_l)D^{(l)}\Big)^p$ is a linear combination of $D^{(h)}$ and $D^{(l)p}$ with coefficients in $k(\Gamma_1\times\cdots\times\Gamma_g)$ and the other terms $D^{(l_1)}\cdots D^{(l_r)}$ $(2\leq r\leq p-1)$ disappear in the expansion. By virtue of (6) α_{lj} is a function only on s_j, hence the coefficients of $D^{(h)}$ and $D^{(l)p}$ in $D_{s_i}^p$ are respectively $D_{s_j}^{p-1}(\alpha_{lj})$ and α_{lj}^p. This proves (10).

Secondly we shall notice:

$$(11)\qquad D_{s_j}^{p}=0,\quad (1\leq j\leq g),$$

since $s_1,\cdots,s_g$ are independent elements in $k(\Gamma_1\times\cdots\times\Gamma_g)$. Hence from (10) we have

$$(12)\qquad -\sum_{h=1}^{g}D_{s_j}^{p-1}(\alpha_{hj})D^{(h)})=\sum_{k=1}^{g}\alpha_{hj}^{p}D^{(h)p},\ (1\leq j\leq g).$$

Operating Ω_i on the both sides of (12), we have

$$-\sum_{h=1}^{g}D_{s_j}^{p-1}(\alpha_{hj})\Omega_i(D^{(h)})=\sum_{h=1}^{g}\alpha_{hj}^{p}\Omega_i(D^{(h)p})$$

and

$$(13)\qquad -D_{s_j}^{p-1}(\alpha_{ij})=\sum_{h=1}^{g}\alpha_{hj}^{p}\Omega_i(D^{(h)p}),\ (1\leq i, j\leq g).$$

Hence from (6), comparing the constant terms of the both sides of (13), we have

$$-(p-1)!\,a_{ij}^{(p-1)}=\sum_{h=1}^{g} a_{hj}^{(0)p}\,\Omega_i(D^{(h)p}) \quad (1\leq i,j\leq g). \tag{14}$$

Since $(p-1)!\equiv -1 \mod p$, (14) shows

$$(\Omega_i(D^{(j)p}))=A^{(p-1)}(A^{(0)\mathfrak{P}})^{-1}=A^{(0)}A^{(0)-1}A^{(p-1)}(A^{(0)\mathfrak{P}}),$$

where $A^{(0)\mathfrak{P}}$ means the matrix whose (i,j)-element is $a_{ij}^{(0)p}$. This shows that $(\Omega_i(D^{(j)}p))$ is the Hasse-Witt matrix of Γ.

Reference

[1] H. Hasse und E. Witt: Zyklische unverzweigte Erweiterungskörper von Primzahlgrade p über einem algebraischen Funktionenkörper der Charakteristik p, Mh. Math. Phs., **43**, 477–492 (1936).