75. *On Distributions and Spaces of Sequences. I*

General Theory

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1. P. Dirac [1] introduced improper function δ when he constructed his theory of representation in quantum mechanics. Further L. Schwartz [2] gave a mathematical foundation of this function using his theory of distributions.

Improper functions in quantum mechanics whose singularities are like δ, thus, can be treated in a mathematically rigorous way.

In quantum field theory, however, not the single δ function but the multiplications of improper functions of same sort of singularity appear [4] and it is widely recognized that the difficulties of the present quantum field theory originate from these very multiplications.

The multiplication of δ type functions are considered by some authors [5] [6] [7] [8]. H. König constructed algebraically the multiplication using tensor product, but he dared not extend his consideration to deeper investigation of the relation between the extended multiplication and the ordinary multiplication defined by L. Schwartz.

E. Stueckelberg and A. Petermann considered the multiplication accompanied with divison and carry out his calculation using these products. W. Güttinger defined the multiplication by the same sort of calculation, but considered it by the way of extension of the research of König. Both of these difinitions are not perfectly rigorous. In addition to these defects, all of three definitions lack in associativity and commutativity, which is very inconvenient for the application in field theory. For example, we can not attain always unique results in calculation of the vertex function [12].

Since the multiplication of two arbitrary distributions is not always a distribution we shall need to extend the notion of distribution in order to study the multiplication of general non-function distributions.

In this and the following papers we extend the notion of distribution to the spaces of sequences and introduce the generalized distribution and consider various operations including the multiplication in these spaces, thus try to construct the new formalism for calculation of some sorts of singular improper functions.

L. Schwartz [9] remarked the nonexistence of the vector space E which satisfies the following conditions:

(1) E contains the space F, where F is a space of all continuous functions. The multiplication coincides on F with usual multiplication.

(2) The multiplication operation is bilinear and associative.

(3) 1 belongs to E, where $1 \cdot T = T$ for all $T \in E$.

(4) x^{-1} belongs to E, where x^{-1} is defined by $x^{-1}\ x=1$.

(5) δ belongs to E, where $\delta \neq 0$ and $x \cdot \delta = 0$.

In the space of generalized distributions, there exists an element 1 which satisfies $1 \cdot T = T$ for any generalized distribution T, and there exist also elements which correspond to distributions $pf\ x^{-1}$, δ each. Apparently this contradicts to L. Schwartz's remark.

The multiplication defined in spaces of sequences satisfies the conditions (1), (2), (3). The conditions (4), (5) are satisfied in the sense of distributions in the spaces of sequences. Of course it is not disirable that the meaning of multiplication differs in cases. But in spite of this odd meaning, our multiplication has possibility to be applied to the field theory.

The defect of theory of spaces of sequences is the indeterminateness of some operations in the spaces. However, so far as multiplications concern, this state of affairs has conversely advantage such that the indeterminateness of multiplication can be associated with the so-called "ambiguities" of current field theory or with renormalization methods [7], [8].

For the sake of simplicity we restrict our considerations to the case of one independent real variable in this paper.

2. Let $\mathfrak{D}$ denote the space of C_0^∞ complex valued functions with compact carriers. Let τ be the topology in $\mathfrak{D}$ which is compatible with the linear structure of the space. Let $(\overline{\mathfrak{D}}, \tau)$ denote the vector space which is a closure of $\mathfrak{D}$ in the topology τ. Let $\tau_{(\mathfrak{D}')}$ denote the topology which is introduced in $\mathfrak{D}$ by the topology in the space of distributions $\mathfrak{D}'$.

Then we see that $(\overline{\mathfrak{D}}, \tau_{(\mathfrak{D}')}) = \mathfrak{D}'$[2], that is to say, any distribution T is considered as sequences $\{\varphi_j^T\}$ such that $\varphi_j^T \in \mathfrak{D}$ and $\{\varphi_j^T\}$ converges to T in the topology of $\mathfrak{D}'$.

Now let $\boldsymbol{Q}^{\mathfrak{D}'}$ denote the space of all sequences $[\{\varphi_j^T\} \mid T \in \mathfrak{D}', \varphi_j \in \mathfrak{D}]$. In this space we can define linear operation as follows: $\alpha\{\varphi_j^T\} + \beta\{\varphi_j^S\} = \{\alpha\varphi_j^T + \beta\varphi_j^S\}$. Then we see that $\{\alpha\varphi_j^T + \beta\psi_j^S\} = \{\alpha\varphi_j^T + \beta\psi_j^S)^{\alpha T + \beta S}\}$ and the algebraic linear structure of the space $\boldsymbol{Q}^{\mathfrak{D}'}$ is homomorph to that of $\mathfrak{D}'$.

Let $O_{\mathfrak{D}'}$ denote the set of all the sequences such that $\varphi_j^0 \to 0$.

Then we see that the quotient space $\widetilde{Q}^{\mathfrak{D}'}\equiv \boldsymbol{Q}^{\mathfrak{D}'}/O_{\mathfrak{D}'}$ is algebraic isomorph to the space $\mathfrak{D}'$.

Now we can introduce another topology τ in the space which is compatible with the linear structure of the space $\mathfrak{D}$. In the topology τ, some squences $\{\varphi_j^T\}$ of $Q^{\mathfrak{D}'}$ may converge to the old $T\in\mathfrak{D}'$, and some sequences $\{\varphi_j^T\}$ may converge to another element $\widetilde{T}(\notin\mathfrak{D}')$ (i.e. they make Cauchy sequences in topology τ), and some sequences $\{\varphi_j^T\}$ may diverge. Let $\widetilde{\boldsymbol{Q}}_\tau^{\mathfrak{D}'}$ denote the set of Cauchy sequence and let $\boldsymbol{O}_\tau$ is the set of all the sequence $\{\varphi_j^0\}$ such that $\{\varphi_j^0\}$ converges to 0 in theto pology τ. The quotient (as an additive group) space $\widetilde{\boldsymbol{Q}}_\tau^{\mathfrak{D}'}/\boldsymbol{O}_\tau=\widetilde{Q}_\tau^{\mathfrak{D}'}$ consists of equivalent classes $\mathfrak{c}(T|\tau|\widetilde{T}_\alpha)$, whose definition are given as follows:

Definition. Let $\mathfrak{c}(T|\tau|T_\alpha)$ be a set of sequences $\{\varphi_n^T\}$ such that $\varphi_n^T\to T$ in the topology of $\mathfrak{D}'$ and $\varphi_n^T\to\widetilde{T}_\alpha$ in the topology τ. We call this class a **convergent generalized distribution $\widetilde{\boldsymbol{T}}_\alpha$ of a distribution T** (in topology τ).

In case $\{\varphi_n^T\}$ diverges in the topology τ, similar equivalence classification by $\mathfrak{c}(0|\tau|0)$ is possible. Then we say the class is a **divergent generalized distribution of $\boldsymbol{T}$** and denote the class by $\mathfrak{c}(T|\tau|\infty_\alpha)$. That is to say, $\mathfrak{c}(T|\tau|\infty_\alpha)$ consists of sequences $\{\varphi_{\alpha,n}^T\}$, $\{\psi_{\alpha,n}^T\}$ such that $\varphi_{\alpha,n}^T\to T, \psi_{\alpha,n}^T\to T$ in $\tau_{\mathfrak{D}'}$ and $\varphi_{\alpha,n}^T\to\infty, \psi_{\alpha,n}^T\to\infty$ in τ and $\varphi_{\alpha,n}^T-\psi_{\alpha,n}^T\in\mathfrak{c}(0|T|0)$.

Now we see the following

Theorem 1. *Generalized distribution are homogenous with respect to every distribution, i.e.* $\bigcup_\alpha\mathfrak{c}(T|\tau|\widetilde{T}_\alpha)$ *can be mapped (algebraically) isomorphic to* $\bigcup_\alpha\mathfrak{c}(0|\tau|\widetilde{0}_\alpha)$.

Proof. Let $\widetilde{T}$ and $\widetilde{T}_0$ be a converging generalized distribution of T in topology τ, then it is evident that $\widetilde{T}-\widetilde{T}_0$ is a converging generalized distribution of 0, i.e., $\widetilde{T}-\widetilde{T}_0=\widetilde{0}$.

Conversely $\widetilde{0}+\widetilde{T}_0$ is an element of $\mathfrak{c}(T|\tau|\widetilde{T}_0)$.

Hence the mapping $\widetilde{T}\leftrightarrow\widetilde{0}(=\widetilde{T}-\widetilde{T}_0)$ is isomorphic for a fixed $\widetilde{T}_0$. Now let $\{\varphi^T\}\in\mathfrak{c}(T|\tau|\infty_\alpha)$, then $\{\varphi^T\}-\widetilde{T}_0\in\mathfrak{c}(0|\tau|\infty_\beta)$, and conversely for any $\{\varphi^0\}\in\mathfrak{c}(0|\tau|\infty_\beta)$, $\{\varphi^0\}+\widetilde{T}_0\in\mathfrak{c}(T|\tau|\infty_\alpha)$.

Now we see some examples of converging generalized distribution.

Example 1. The following various spaces introduced by L. Schwartz can be identified with our spaces of sequences.

$$\begin{array}{ccccccccccc}
\mathfrak{D} & \subset & \mathfrak{S} & \subset & \mathfrak{D}_{L^p} & \underset{1\le p<q<\infty}{\subset} & \mathfrak{D}_{L^q} & \subset & \dot{\mathfrak{B}} & & \subset \mathcal{E} \\
\cap & & & & \cap & & \cap & & & & \cap \\
\mathcal{E}'\subset & & & & \mathfrak{D}'_{L^p} & \subset & \mathfrak{D}'_{L^q} & \subset & \dot{\mathfrak{B}}' & \subset\mathfrak{S}'\subset & \mathfrak{D}'
\end{array}$$

i.e. $\tilde{Q}^{\mathfrak{D}'}_{\tau_{\mathfrak{S}}}=\mathfrak{S}$, $\tilde{Q}^{\mathfrak{D}'}_{\tau_{\mathfrak{S}'}}=\mathfrak{S}'$ etc.
Especially for the discrete topology τ_d, $\tilde{Q}^{\mathfrak{D}'}_{\tau_d}=\mathfrak{D}$. In these examples set of converging generalized distribution $\bigcup_\alpha \mathfrak{c}(0|\tau|\tilde{0}_\alpha)$ consist of only one element $\mathfrak{c}(0|\tau|0)$. Hence by Theorem 1 we can see that $\bigcup_\alpha \mathfrak{c}(T|\tau|\tilde{T}_\alpha)$ consists of one elment $\tilde{T}_\alpha=T$ and in case $T\notin Q_\tau$, $\{\varphi_j^T\}$ diverges i.e. $\{\varphi_j^T\}\in \mathfrak{c}(T|\tau|\infty_\alpha)$.

The following spaces are outside of above diagram, and is one of the object of our main interest in this and the following papers.

Example 2. The topology τ is defined by the following types of neighborhood $\boldsymbol{U}^n$ of zero:

$$\boldsymbol{U}^n_{\alpha.W}(0)=\{\varphi|\quad (\varphi(x))^\alpha\in W_{\mathfrak{D}'}(0)\},$$

where $W_{\mathfrak{D}'}(0)$ means a neighbourhood of zero in the space $\mathfrak{D}'$, and $\alpha=2m+1$ for positive integer. We see easily that the systems $\{\boldsymbol{U}^n(0)\}$ such that the corresponding $W_{\mathfrak{D}'}(0)$ runs through all the neighborhood system of zero in the topology $\mathfrak{D}'$, defines topology τ which is compatible with linear operations of $\mathfrak{D}$. We denote this topology by $\tau_{1/\alpha}$ and denote the space of converging generalized distribution by $\tilde{Q}^{\mathfrak{D}'}_{1/\alpha}$.

Now we define $\delta^{1/\alpha}$ for $\alpha=2m+1$. Let ρ_n denote the function

$$\rho_n(x)=\begin{cases} 0 \quad \text{for} \quad |x|\geq 1/n^n \\ kn^n\ \exp\{1/(n^{2n}x^2-1)\} \text{ for } |x|<1/n^n, \end{cases}$$

where the constant k satisfies the equality

$$k\int_{-1}^{1}\exp\{1/(x^2-1)\}dx=1.$$

The sequence $\{(\rho_n(x))^{1/\alpha}\}$ converges to 0 in $\mathfrak{D}'$ and converges to 0 in $\tau_{1/\beta}$ for $\alpha>\beta>1$ and diverges in $\tau_{1/\gamma}$ for $\gamma>\alpha$, where α,β,γ are positive integers. In the topology $\tau_{1/\alpha}$ the sequence makes a Cauchy sequence. We call this sequence $\delta^{1/\alpha}$ or $\delta^{1/\alpha}(0)$. By translation in the x-space we obtain $\delta^{1/\alpha}(x_0)$ quite similarly.

Using these notations, for example above properties are stated as following:

$$Q^{\mathfrak{D}'}_{1/3}\ni \mathfrak{c}(0|\tau_{1/3}|\tilde{0}_\alpha)\ni\delta^{1/3}(x_0) \text{ for any } x_0,$$
$$Q^{\mathfrak{D}'}_{1/5}\ni \mathfrak{c}(0|\tau_{1/5}|\infty_\beta)\ni\delta^{1/3}(x_0) \text{ for any } x_0.$$

$\delta^{1/5}(x_0)$ belongs to $\mathfrak{c}(0|\tau_{1/3}|\tilde{0}_\alpha)\in Q^{\mathfrak{D}'}_{1/3}$ also, and in this space $\delta^{1/5}(x_0)$ is equivalent to zero element. We can see easily also

$$\delta\in \mathfrak{c}(\delta|\tau_{1/3}|\infty)\in \boldsymbol{Q}^{\mathfrak{D}'}_{1/3}.$$

3. We sketch here briefly properties of multiplicative operation. Detailed properties is discussed in II [10]. Other operations such as derivation, convolution and the Fourier Transform will be discussed in III.

We define the product of sequences $\{\varphi_j^S\}$ and $\{\varphi_j^T\}$ by $\{\varphi_j^S\cdot\varphi_j^T\}$, and define the product (1) of converging or diverging generalized distribution of $\mathfrak{c}(T|\tau_1|\ )$ and $\mathfrak{c}(S|\tau_2|\ )$ by sequences $\{\{\psi_n\}|\psi_n=\varphi_n^T\cdot\varphi_n^S, \{\varphi_n^T\}\in \mathfrak{c}(T|\tau_1|\ ), \{\varphi_n^S\}\in\mathfrak{c}(S|\tau_2|\ )\}$.

In order to classify the product, we introduce the following definitions. Let $\boldsymbol{Q}$ denote the space of all the sequences $\{\varphi_n\}$ of C^∞ functions φ_n (with arbitrary carriers).

We have hitherto constructed the space Q_τ by the sequence $\{\varphi_j\}$ of $\varphi_j\in\mathfrak{D}$, but similar construction is possible using sequence $\{\varphi_j\}$ of $\varphi_j\in\mathcal{E}$ by slight modifications of definitions. Let Q_τ be the space of all classes of type $\mathfrak{c}(\ |\tau|\widetilde{T}_\alpha)$ and $\mathfrak{c}(\ |\tau|\infty_\beta)$.

Let P_τ denote a natural mapping from the space $\boldsymbol{Q}$ onto Q_τ.

We define the product (τ_3) of two classes $\mathfrak{c}(T|\tau_1|\ )\in Q_{\tau_1}$, and $\mathfrak{c}(s|\tau_2|\ )\in Q_{\tau_2}$ by $[\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(s|\tau_2|\ )]_{\tau_3}=P_\tau\{\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(s|\tau_2|\ )\}$.
The ordinary multiplication of distribution is defined only between regular functions and distributions T for example $\alpha\in\mathcal{E}, T\in\mathfrak{D}'$. These original definitions correspond to a part of our multiplication. For example, the multiplication between $\alpha\in\mathcal{E}$ and $T\in\mathfrak{D}'$ corresponds to the case of our products in which τ_1 is the discrete topology in $\mathcal{E}$ and $\tau_2=\tau_3=\tau_{\mathfrak{D}'}$.

Remark. L. Schwartz showed the following theorem [9]:

Let E be a vector space on real numbers, whose subspace is the space F of all continuous function of one real variable.

The multiplication which satisfies the following condition can not exist:

(1) bilinear associative operation,

(2) coincides on F with usual multiplication,

(3) $1\in E$ where $1\cdot e=e$ for any $e\in E$,

(4) $x^{-1}\in E$ where $x^{-1}x=1$,

(5) $\delta\in E$ where $x\cdot\delta=0$, $\delta\neq 0$.

Now in space $\boldsymbol{Q}$ we introduce topology τ_0 by the ordinary point-wise convergence as a sequence of function. Then $\widetilde{\boldsymbol{Q}}_{\tau_0}$ satisfy the conditions (1), (2), and (3).

Since there are no δ and x^{-1} in $\widetilde{\boldsymbol{Q}}_{\tau_0}$, the conditions (4) and (5) are not satisfied. However, if we change the classification using $\mathfrak{D}'$ topology, then (4), (5) are satisfied in the sense of distributions.

This theorem does not exclude the possible existence of a multiplication which is associative and commutative and satisfies law of usual multiplication for continuous function and usual law of distribution product for pfx^{-1} or δ. The different meaning of two sort of multiplications will not be an obstacle for the application to the field theory.

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