143. Some Characterizations of $\mathfrak{m}$-paracompact Spaces. II

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In this paper we study some characterizations of $\mathfrak{m}$-paracompact and normal spaces in the form of the selection theorems.[1] Let X and Y be topological spaces. 2^Y will denote the family of non-empty subsets of Y. A function from a subset of X to 2^Y is called a *carrier*. If φ: $X\to 2^Y$, then a *selection* for φ is a continuous function $f: X\to Y$ such that $f(x)\in\varphi(x)$ for every $x\in X$. A carrier φ: $X\to 2^Y$ is *lower semi-continuous* if, whenever $V\subset Y$ is open in Y, $\{x\in X\mid\varphi(x)\cap V\neq\phi\}$ is open in X, where ϕ denotes the null set. For a Banach space or a complete metric space Y, we shall consider the following families of sets.

$$A(Y)=\{S\in 2^Y\mid S \text{ is closed}\},$$
$$K(Y)=\{S\in 2^Y\mid S \text{ is convex}\},$$
$$F(Y)=\{S\in K(Y)\mid S \text{ is closed}\},$$
$$C(Y)=\{S\in F(Y)\mid S \text{ is compact or } S=Y\}.$$

The following theorem seems to be interesting for us in the point of view that Michael's results [3, Theorems 3.1″ and 3.2″], which were separately stated and proved for paracompact spaces and countably paracompact spaces, are unified.

Theorem 1. *The following properties of a T_1-space are equivalent.*

(a) *X is $\mathfrak{m}$-paracompact and normal.*

(b) *If Y is a Banach space which has an open base of power $\leqq\mathfrak{m}$, then every lower semi-continuous carrier $\varphi: X\to F(Y)$ admits a selection.*

To prove this theorem, the following lemmas and Theorem 2 in the previous paper [2] are useful.

Lemma 1. *If X is $\mathfrak{m}$-paracompact and normal, Y a normed linear space with an open base of power $\leqq\mathfrak{m}$, $\varphi: X\to K(Y)$ a lower semi-continuous carrier, and if V is a convex neighborhood of the origin of Y, then there exists a continuous function $f: X\to Y$ such that $f(x)\in\varphi(x)+V$ for every x in X.*

Proof. Since $\{y-V\}_{y\in Y}$ is an open covering of Y and Y has an open base with power $\leqq\mathfrak{m}$, there exists a locally finite open refinement $\{W_\lambda\mid\lambda\in\Lambda\}$ of $\{y-V\}_{y\in Y}$ with $|\Lambda|\leqq\mathfrak{m}$. Let $U_\lambda=\{x\in X\mid\varphi(x)\cap W_\lambda$

1) Cf. E. Michael [3].

$\neq \phi\}$. Then, by the definition of a lower semi-continuous carrier, U_λ is open in X, and clearly $\mathfrak{U}=\{U_\lambda \mid \lambda \in \Lambda\}$ is an open covering of X. Since X is $\mathfrak{m}$-paracompact and normal, there exists a locally finite partition of unity $P=\{p_\alpha \mid \alpha \in \Omega\}$ on X subordinated to $\mathfrak{U}$, with $|\Omega| \leqq \mathfrak{m}$. Now for each $\alpha \in \Omega$, pick a $\lambda(\alpha) \in \Lambda$ and a $y_\alpha \in Y$ such that p_α vanishes outside $U_{\lambda(\alpha)} \in \mathfrak{U}$ and $W_{\lambda(\alpha)} \subset y_\alpha - V$. We can now set

$$f(x)=\sum p_\alpha(x) y_\alpha.$$

Then it is obvious that $f(x)$ is a continuous function of X into Y. Since, for any $x_0 \in X$, $\{\alpha \in \Omega \mid p_\alpha(x_0)>0\}$ is a finite subset of Ω, we denote it by $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then, since $x_0 \in \{x \mid p_{\alpha_i}(x)>0\} \subset U_{\lambda(\alpha_i)}$, we obtain $\varphi(x_0) \cap W_{\lambda(\alpha_i)} \neq \phi$. Hence it follows from $W_{\lambda(\alpha_i)} \subset y_{\lambda(\alpha_i)} - V$ that $\varphi(x_0) \cap (y_{\lambda(\alpha_i)} - V) \neq \phi$. Thus we have

$$y_{\lambda(\alpha_i)} \subset \varphi(x_0) + V \quad (i=1, 2, \cdots, n),$$

which means that

$$f(x_0)=\sum p_\alpha(x_0) y_\alpha \in \varphi(x_0) + V.$$

This completes the proof of this lemma.

Lemma 2. ([3, Theorem 3.1′]) *The following properties of a T_1-space are equivalent.*

(a) *X is normal.*

(b) *If Y is a separable Banach space, then every lower semi-continuous carrier $\varphi: X \to C(Y)$ admits a selection.*

Proof of Theorem 1. (a)$\to$(b). This can be proved by the same way as in the proof of [3, Theorem 3.2″] besides using Lemma 1. (b)$\to$(a). From the same arguments as in the proof of [3, Theorem 3.2″], if follows that every open covering $\mathfrak{U}=\{U_\lambda \mid \lambda \in \Lambda\}$ of X with $|\Lambda| \leqq m$ admits a partition of unity (not necessarily locally finite) subordinated to it. Since it follows from Lemma 2 that X is normal, and X is $\mathfrak{m}$-paracompact by virtue of Theorem 2 in the previous paper [2], we complete the proof.

Corollary 1. ([3, Theorem 3.1″]) *The following properties of a T_1-space are equivalent.*

(a) *X is normal and countably paracompact.*

(b) *If Y is a separable Banach space, then every lower semi-continuous carrier $\varphi: X \to F(Y)$ admits a selection.*

Corollary 2. ([3, Theorem 3.2″]) *The following properties of a T_1-space are equivalent.*

(a) *X is paracompact.*

(b) *If Y is a Banach space, then every lower semi-continuous carrier $\varphi: X \to F(Y)$ admits a selection.*

In the sequel we characterize a 0-dimensional $\mathfrak{m}$-paracompact and normal space by the property of lower semi-continuous carriers.

Theorem 2. *The following properties of a T_1-space are equivalent.*

(a) *X is 0-dimensional* $\mathfrak{m}$*-paracompact and normal.*

(b) *If Y is a Banach space which has an open base of power* $\leqq \mathfrak{m}$*, then every lower semi-continuous carrier* $\varphi: X \to A(Y)$ *admits a selection.*

As a first step, we shall state the following lemmas.

Lemma 3. ([5, Theorem 2.1]) *If X is a normal space of dimension* $\leqq n$*, then any locally finite open covering of X has an open refinement of order* $\leqq n+1$.

Lemma 4. *If X is 0-dimensional* $\mathfrak{m}$*-paracompact and normal, Y a paracompact uniform space with an open base of power* $\leqq \mathfrak{m}$*,* $\varphi: X \to 2^Y$ *a lower semi-continuous carrier, and if V is a symmetric uniform neighborhood of Y, then there exists a continuous function* $f: X \to Y$ *such that* $f(x) \in V(\varphi(x))$ *for every x in X.*

Proof. Since $\{V(y) \mid y \in Y\}$ is an open covering of Y and Y has an open base with power $\leqq \mathfrak{m}$, there exists a locally fiinite open refinement $\{W_\lambda \mid \lambda \in \Lambda\}$ of $\{V(y) \mid y \in Y\}$, with $|\Lambda| \leqq \mathfrak{m}$. Let $U_\lambda = \{x \in X \mid \varphi(x) \cap W_\lambda \neq \phi\}$. Clearly U_λ is open in X and $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ is an open covering of X. Since X is 0-dimensional $\mathfrak{m}$-paracompact and normal, it follows from Lemma 3 that there exists an open refinement $\mathfrak{V} = \{V_\alpha \mid \alpha \in \Omega\}$ of $\mathfrak{U}$ such that $V_\alpha \cap V_\beta = \phi$ as $\alpha \neq \beta$. Now, for each $\alpha \in \Omega$, pick a $\lambda(\alpha) \in \Lambda$ and a y_α in Y such that $V_\alpha \subset U_{\lambda(\alpha)}$ and $W_{\lambda(\alpha)} \subset V(y_\alpha)$. Since, for any x in X, there exists only one element $\alpha(x) \in \Omega$ such that $x \in V_{\alpha(x)}$. We now set $f(x) = y_\alpha$ as $x \in V_\alpha$. Then it is obvious that $f(x)$ is a continuous function of X into Y. Since $x \in V_\alpha$ implies $x \in U_{\lambda(\alpha)}$, we obtain $\varphi(x) \cap W_{\lambda(\alpha)} \neq \phi$. Hence $\varphi(x) \cap V(y_\alpha) \neq \phi$. This shows that $y_\alpha \in V(\varphi(x))$. Thus $f(x)$ satisfies all our requirements. This completes the proof of this lemma.

Proof of Theorem 2. Since we can prove that (a)→(b) by the same way as in the proof of [3, Theorem 3.2″] without any alterations besides using Lemma 4, we shall show only that (b)→(a).

Let $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be an open covering of X with $|\Lambda| \leqq \mathfrak{m}$, and let us consider Λ as a metric space such that each pair of distinct points of Λ have distance 1. Then a metric space Λ can be imbedded as a neighborhood retract in a suitable (generalized) Hilbert space Y with an open base of power $\leqq \mathfrak{m}$.[2] Now, for any $x \in X$, let $\varphi(x) = \{\lambda \mid x \in U_\lambda\}$. Then $\varphi: X \to A(Y)$ is lower semi-continuous. In fact, for any subset Λ_0 of Λ, $\{x \in X \mid \varphi(x) \cap \Lambda_0 \neq \phi\} = \bigcup_{\lambda \in \Lambda_0} U_\lambda$, which shows lower semi-continuity of φ. Hence, by (*b*), we can select a continuous function f: $X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$. Since the inverse images $f^{-1}(\lambda)$ form an open covering of X of order 1 which refines $\mathfrak{U}$, X becomes a 0-dimensional $\mathfrak{m}$-paracompact space. Furthermore it

2) Cf. Dowker [1, Remark (p. 313)].

follows from Lemma 2 that X is normal. This completes the proof.

Corollary 1. *The following properties of a T_1-space are equivalent.*

(a) *X is 0-dimensional countably paracompact and normal.*

(b) *If Y is a separable Banach space, then every lower semi-continuous carrier $\varphi: X \to A(Y)$ admits a selection.*

Corollary 2. *The following properties of a T_1-space are equivalent.*

(a) *X is 0-dimensional paracompact and normal.*

(b) *If Y is a Banach space, then every lower semi-continuous carrier $\varphi: X \to A(Y)$ admits a selection.*

Remark. As is easily seen, we can replace "a Banach space" in Theorem 2 and Corollaries 1, 2 with "a complete metric space".

Corollary 3. *If Y is a complete metric space with an open base of power $\leqq \mathfrak{m}$, X a 0-dimensional $\mathfrak{m}$-paracompact and normal space, and if the map $u: Y \to X$ is continuous, open and onto, then there exists a continuous $f: X \to Y$ such that $f(x) \in u^{-1}(x)$ for every $x \in X$.*

Proof. Define $\varphi: X \to A(Y)$ by $\varphi(x) = u^{-1}(x)$. By Example 1.1* of [3], φ is lower semi-continuous. Hence, by Theorem 2, there exists a selection f for φ, and clearly f satisfies our requirements.

[4, Corollary 1.4] is an immediate consequence of this corollary.

References

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