155. *Local Times on the Boundary for Multi-Dimensional Reflecting Diffusion*

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1. In connection with the study of the multi-dimensional diffusion processes with general boundary conditions, T. Ueno [8] introduced the notion of *the Markov process on the boundary concerning the diffusion.* He made a conjecture that this process would be obtained by a suitable change of time scale from the diffusion. As a typical case, for the reflecting Brownian motion on the half space he defined the local time on the boundary and showed that this is suitable as the time change function.

In this paper we shall define the local times on the boundary for the reflecting diffusion in multi-dimensions. This is the generalization of the local time at a single point for one-dimensional diffusion defined by P. Lévy [6], H. Trotter [7] and K. Itô-H. P. McKean, Jr. [3]. The results were published in [2] in mimeographed form with detailed proofs. Applications to the Markov processes on the boundary and diffusions with more general boundary conditions will appear in a paper by the first author.

We would like to thank N. Ikeda and T. Ueno. Our work owes much to them.

2. **The reflecting diffusion.** Let D be a domain with compact closure $\overline{D}$ in an N-dimensional orientable manifold of class C^∞. Assume that D has non-empty boundary ∂D consisting of a finite number of components each of which is an $(N-1)$-dimensional hypersurface of class C^3. We denote the local coordinate of the point x as $(x^1,\cdots,x^N)$. Let A be a second-order elliptic differential operator:

$$(2.1)\qquad Au(x)=\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\frac{\partial u(x)}{\partial x^j}\right)+b^i(x)\frac{\partial u}{\partial x^i},$$

where $a^{ij}(x)$ and $b^i(x)$ are contravariant tensors on $\overline{D}$ of class C^3, $a^{ij}(x)$ is symmetric and strictly positive definite for each $x\in\overline{D}$, and $a(x)=\det(a^{ij}(x))^{-1}$. In (2.1) we used the summation convention in differential geometry and $Au(x)$ is independent of the choice of local coordinates. The (inner) normal derivative is defined as

$$(2.2)\qquad \frac{\partial u(x)}{\partial n}=\frac{1}{\sqrt{a^{NN}(x)}}a^{Ni}(x)\frac{\partial u(x)}{\partial x^i},\qquad x\in\partial D,$$

when in a neighborhood of x

(2.3) ∂D is represented as $x^N=0$, and D as $x^N>0$.
$m(dx)$ and $\tilde{m}(dx)$ are the Riemannian volume and surface elements respectively, that is,

$$m(E)=\int_E \sqrt{a(x)}\,dx^1\cdots dx^N, \qquad E\subset \overline{D}, \tag{2.4}$$

and when (2.3) is satisfied

$$\tilde{m}(E)=\int_E \sqrt{a(x)}\sqrt{a^{NN}(x)}\,dx^1\cdots dx^{N-1}, \qquad E\subset \partial D. \tag{2.5}$$

$\dfrac{\partial}{\partial n}$, m and $\tilde{m}$ are also independent of the choice of local coordinates.

Our assumptions satisfy the conditions in S. Itô [4], so that there exists a unique fundamental solution $p(t, x, y)$ for the initial value problem of the equation

$$\frac{\partial u(t, x)}{\partial t}=Au(t, x) \tag{2.6}$$

with the boundary condition

$$\frac{\partial u(t, x)}{\partial n}=0. \tag{2.7}$$

$p(t, x, y)m(dy)$ has the property of Markov transition measure and moreover we can prove that

$$\lim_{t\downarrow 0}\sup_{x\in\overline{D}}\frac{1}{t}\Big[1-\int_{U_\varepsilon(X)} p(t, x, y)m(dy)\Big]=0 \tag{2.8}$$

where $U_\varepsilon(x)$ is the ε-neighborhood of x by a metric giving the topology on $\overline{D}$. Accordingly, by the theorem of Dynkin [1] and Kinney [5], we can construct a continuous conservative Markov process $[x_t, W, \boldsymbol{B}, P_x]$ with transition probability $p(t, x, y)m(dy)$, which we call *the reflection A-diffusion*. Here W is the space of continuous paths on $\overline{D}$, $x_t(w)$ is the coordinate function and P_x is the probability measure starting at x over the coordinate Borel field $\boldsymbol{B}$.

3. **Definition of the local time on the boundary** as an application of a theorem on excessive functions.

For $\alpha>0$, define

$$p_\alpha(x)=\int_0^\infty e^{-\alpha t}dt\int_{\partial D} p(t, x, y)\tilde{m}(dy), \tag{3.1 a}$$

$$p_\alpha(t, x)=e^{-\alpha t}\int_{\overline{D}} p(t, x, y)p_\alpha(y)m(dy)=E_x[e^{-\alpha t}p_\alpha(x_t)].^{*)} \tag{3.1 b}$$

From the non-negativity and Chapman-Kolmogorov equation for $p(t, x, y)$ and from the fact that $p_\alpha(x)$ is continuous on $\overline{D}$ by a result of S. Itô [4], it follows that

(3.2 a) $p_\alpha(x)$ is non-negative and bounded,

*) E_x denotes expectation with respect to the probability measure P_x.

(3.2 b) $p_\alpha(t, x) \uparrow p_\alpha(x)$ as $t \downarrow 0$ uniformly on $\overline{D}$,

(3.2 c) $p_\alpha(t, x) \downarrow 0$ as $t \uparrow \infty$.

By a theorem of V. A. Volkonski [9, 10] (or see [2]), (3.2) implies that *there is a unique continuous non-negative functional* $\mathfrak{t}^\alpha(t, w)$ *which is additive in the sense that*

$$\mathfrak{t}^\alpha(t, w) = \mathfrak{t}^\alpha(s, w) + e^{-\alpha s}\mathfrak{t}^\alpha(t-s, w_s^+), \qquad t \geq s \geq 0,$$

and with the expectation

$$E_x[\mathfrak{t}^\alpha(\infty, w)] = p_\alpha(x), \tag{3.3}$$

where w_s^+ is the shifted path $w_s^+ : t \in [0, \infty) \to x_{t+s}(w)$. Put

$$\mathfrak{t}(t, w) = \int_0^t e^{\alpha s}\mathfrak{t}^\alpha(ds, w). \tag{3.4}$$

Then we can prove easily the following

THEOREME 1. $\mathfrak{t}(t, w)$ *is a continuous non-negative additive functional:*

(3.5 a) *for each* $t \geq 0$ $\mathfrak{t}(t, w)$ *is measurable on the sample space* $[x_s : s \leq t]$,

(3.5 b) $0 = \mathfrak{t}(0, w) \leq \mathfrak{t}(t, w)$ *and* $\mathfrak{t}(t, w)$ *is continuous in* t,

(3.5 c) $\mathfrak{t}(t, w) = \mathfrak{t}(s, w) + \mathfrak{t}(t-s, w_s^+)$, $t \geq s \geq 0$, *and satisfies*

$$E_x[\mathfrak{t}(t, w)] = \int_0^t ds \int_{\partial D} p(s, x, y)\tilde{m}(dy); \tag{3.6}$$

$\mathfrak{t}$ *is flat on each time interval in which* x *is inside* D. *Such an additive functional is unique up to* P_x *measure zero for any* $x \in \overline{D}$. *Especially,* $\mathfrak{t}(t, w)$ *is independent of* α.

DEFINITION. We call $\mathfrak{t}(t, w)$ *the local time on the boundary for the reflecting A-diffusion.*

4. Another definition of the local time on the boundary can be made as the limit of sojourn times on boundary strips. First we define a distance by a^{ij}. If C is a continuous and piecewise smooth mapping:

$$\lambda \in [0, 1] \to x(\lambda) = (x^1(\lambda), \cdots, x^N(\lambda)) \in \overline{D},$$

we define

$$l(\mathrm{C}) = \int_0^1 \left(a_{ij}(x(\lambda)) \frac{dx^i(\lambda)}{d\lambda} \frac{dx^j(\lambda)}{d\lambda} \right)^{\frac{1}{2}} d\lambda$$

where $(a_{ij}(x))$ is the inverse matrix (covariant tensor) of $a^{ij}(x)$. For any $x, y \in \overline{D}$, define a distance $d(x, y)$ between x and y as the infimum of $l(\mathrm{C})$ of all such C from x to y. $d(x, \partial D)$ denotes the infimum of $d(x, y)$ for all $y \in \partial D$. Using this distance we define the boundary strip $D_\rho = \{x \in \overline{D}; d(x, \partial D) < \rho\}$. We can prove the follwing

LEMMA. *For any continuous function* f *on* $\overline{D}$,

$$\lim_{\rho \downarrow 0} \frac{1}{\rho} \int_{D_\rho} f(x) m(dx) = \int_{\partial D} f(x)\tilde{m}(dx). \tag{4.1}$$

If $f = f_\alpha$ *and* $\{f_\alpha\}$ *is a family of equi-continuous and uniformly* *bounded functions on* $\overline{D}$, (4.1) *holds uniformly in* α.

Making use of this lemma, the following facts are shown. Put

$$\mathfrak{t}_\rho(t, w) = \frac{1}{\rho}\int_0^t \chi_{D_\rho}(x_s(w))ds, \tag{4.2}$$

$$e_\rho(t, x) = E_x[\mathfrak{t}_\rho(t, w)] = \frac{1}{\rho}\int_0^t ds \int_{D_\rho} p(s, x, y)m(dy), \tag{4.3}$$

where χ_{D_ρ} is the indicator function of the set D_ρ. Then, we have

$$\lim_{\rho \downarrow 0} e_\rho(t, x) = \int_0^t ds \int_{\partial D} p(s, x, y)\widetilde{m}(dy) = E_x[\mathfrak{t}(t, w)], \tag{4.4}$$

and the following

THEOREM 2.

$$E_x[|\mathfrak{t}_\rho(t, w) - \mathfrak{t}(t, w)|^2] \to 0, \quad \rho \to 0 \tag{4.5}$$

uniformly in $x \in \overline{D}$. *Moreover, there is a sequence* $\rho_n \downarrow 0$ *such that for all* x

(4.6) $P_x(\lim_{n\to\infty} \mathfrak{t}_{\rho_n}(t, w) = \mathfrak{t}(t, w)$ *uniformly on any finite time interval*) $=1$.

Here $\mathfrak{t}(t, w)$ *is the local time on the boundary defined in* 3.

By this theorem we can also *define* the local time on the boundary as $\lim \mathfrak{t}_{\rho_n}(t, w)$. This is the analogue of a definition of the local time for one-dimensional diffusion by H. Trotter [7] and K. Itô-H. P. McKean, Jr. [3].

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