3. On the Existence and the Propagation of Regularity of the Solutions for Partial Differential Equations. I

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1. Introduction. The object of this note is to derive a priori inequality based on our recent note [4], which is applicable to the existence theorem and the propagation of regularity of the solutions for partial differential equations.

Recently L. Hörmander [2] has already derived a similar inequality under some conditions for the principal part of given operators.

We shall consider differential operator L in a neighborhood of the origin in $(\nu+1)$-space: $(t, x)=(t, x_1, \cdots, x_\nu)$. Let $(m, \mathfrak{m})=(m, m_1, \cdots, m_\nu)(m_j \leqq m; j=1, \cdots, \nu)$ be an appropriate real vector whose elements are positive integers. The operator considered in this note is of the form

$$L=L_0+\sum_{i+m|\alpha:\mathfrak{m}|\leqq m-1} b_{i,\alpha}(t, x)\frac{\partial^{i+|\alpha|}}{\partial t^i \partial x^\alpha} \tag{1.1}$$

with

$$L_0=\sum_{i+m|\alpha:\mathfrak{m}|=m} a_{i,\alpha}(t, x)\frac{\partial^{i+|\alpha|}}{\partial t^i \partial x^\alpha} \quad (a_{m,0}(t, x)=1) \tag{1.2}$$

$$(\alpha=(\alpha_1, \cdots, \alpha_\nu),\ x^\alpha=x_1^{\alpha_1}\cdots x_\nu^{\alpha_\nu},\ |\alpha|=\alpha_1+\cdots+\alpha_\nu,$$
$$|\alpha:\mathfrak{m}|=\alpha_1/m_1+\cdots+\alpha_\nu/m_\nu)$$

where $b_{i,\alpha}$ are in L^∞ and $a_{i,\alpha}$ in $C^\infty_{(t,x)}$.[1)]

Setting for (1.2) and real vectors $\xi=(\xi_1, \cdots, \xi_\nu)$

$$L_0(t, x, \lambda, \xi)=\sum_{i+m|\alpha:\mathfrak{m}|=m} a_{i,\alpha}(t, x)\lambda^i \xi^\alpha \tag{1.3}$$

which we call the characteristic polynomial of L, we derive a priori inequality (3.3) under some conditions for the characteristic roots $\lambda=\lambda(\xi)$ of the equation $L_0(t, x, \lambda, \sqrt{-1}\,\xi)=0$ for $\xi \neq 0$.

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2. Definitions and lemmas. Let us define $r=r(\xi)$ for real vector ξ as a positive root of the equation

$$F(r, \xi)\equiv \sum_{j=1}^{\nu} \xi_j^2 r^{-2/m_j}=1 \quad (\xi \neq 0). \tag{2.1}$$

Then, r is in $C^\infty_{(\xi\neq 0)}$ and satisfies inequalities

1) Strictly speaking it is sufficient to assume that $a_{i,\alpha}$ are in $C^k_{(t,x)}$ for $k\geqq m+(\nu+1)\underset{1\leqq j\leqq \nu}{Max}\, m/m_j$.

$$\begin{aligned}(2.2)\qquad &\nu^{-1/2}K(\xi)^m \leqq r(\xi) \leqq \nu^{m/2}K(\xi)^m,\\ &\left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha} r(\xi)\right| \leqq C_\alpha^{2)} K(\xi)^{m(1-|\alpha:\mathfrak{m}|)}\end{aligned}$$

where $K(\xi)$ is defined by $K(\xi)=\left\{\sum_{j=1}^{\nu}\xi_j^{2m_j}\right\}^{1/2m}$.

The proof is not so difficult; see [4].

Definition 1. We call H a singular integral operator of class $C_{\mathfrak{m}}^m$ with the symbol $\sigma(H)(x,\xi)=\sum_{r=1}^{\infty}a_r(x)\hat{h}_r(\xi)$ $(a_r(x)\in C^\infty_{(x)},\ \hat{h}_r(\xi)\in C^\infty_{(\xi\neq 0)};$ $r=1,2,\cdots)$ if the following conditions are satisfied:

$$\begin{aligned}(2.3)\qquad &\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}a_r(x)\right| \leqq A_{l,\alpha}r^{-l} \text{ for every } l \text{ and } \alpha,\\ &\left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha}\hat{h}_r(\xi)\right| \leqq B_\alpha r^{l_\alpha}K(\xi)^{-m|\alpha:\mathfrak{m}|} \text{ for every } \alpha.\end{aligned}$$

Then, Hu is defined by

$$Hu=\frac{1}{\sqrt{2\pi}^{\nu}}\int e^{\sqrt{-1}x\cdot\xi}\sigma(H)(x,\xi)\hat{u}(\xi)d\xi.^{3)}$$

Definition 2. We define a convolution operator Λ by $\widehat{\Lambda u}(\xi)=\hat{\Lambda}(\xi)\hat{u}(\xi)$ where $\hat{\Lambda}(\xi)(\in C^\infty_{(\xi\neq 0)})$ satisfies

$$\left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha}\hat{\Lambda}(\xi)\right| \leqq \gamma_\alpha K(\xi)^{1-m|\alpha:\mathfrak{m}|}.$$

Remark. i) If $\lambda_0(x,\eta)$ $(\in C^\infty_{(x,\eta)}$ $(\eta=(\eta_1,\cdots,\eta_\nu)\neq 0))$ is homogeneous of order zero in η, then by [1] λ_0 is expanded such as $\lambda_0(x,\eta)$ $=\sum_{r=1}^{\infty}a_r(x)\hat{h}_{0,r}(\eta)$ where $a_r(x)$ and $\hat{h}_{0,r}(\eta)$ satisfy (2.3) for $(m,\mathfrak{m})$ $=(1,1,\cdots,1)$ and $K(\eta)=\left\{\sum_{j=1}^{\nu}\eta_j^2\right\}^{1/2}$. Hence, if we define a matrix R by

$$(2.4)\qquad R=\begin{pmatrix} r^{1/m_1} & & 0\\ & \ddots & \\ 0 & & r^{1/m_\nu}\end{pmatrix}$$

and set $\hat{h}_r(\xi)=\hat{h}_{0,r}(\xi R^{-1})$, we can write $\lambda_0(x,\xi R^{-1})=\sum_{r=1}^{\infty}a_r(x)\hat{h}_r(\xi)$. This shows that $\lambda_0(x,\xi R^{-1})$ is the symbol of an operator of class $C_{\mathfrak{m}}^m$. ii) Setting $\hat{\Lambda}=r^{1/m}$ or $\hat{\Lambda}=K(\xi)^{4)}$ we can define an operator Λ.

Lemma 1. *Let $P_i(t)$ and $Q_i(t)$ $(i=1,\cdots,k)$ be in $C_{\mathfrak{m}}^m$ with real valued symbols defined in (x)-space with t as a parameter.*

Suppose each pair $P_i(t)$ and $Q_i(t)$ $(i=1,\cdots,k)$ satisfies the condition of M. Matsumura [5], that is for some $H_i(t)\in C_{\mathfrak{m}}^m$

2) We shall denote by C positive constants, not necessarily the same even in the same formula.

3) For $u\in L^2$ we define the Fourier transform $\mathfrak{F}[u]$ by $\mathfrak{F}[u](\xi)=\hat{u}(\xi)$
$=\frac{1}{\sqrt{2\pi}^{\nu}}\int e^{\sqrt{-1}x\cdot\xi}u(x)\,dx\left(x\cdot\xi=\sum_{j=1}^{\nu}x_j\cdot\xi_j\right)$.

4) In what follows we shall use a notation Λ_0 in the case $\hat{\Lambda}=K(\xi)$.

$$\frac{\partial}{\partial t}\sigma(P_i)+\sum_{j=1}^{\nu}\left\{\frac{\partial}{\partial x_j}\sigma(P_i)\frac{\partial}{\partial \xi_j}(\sigma(Q_i)\hat{\Lambda})-\frac{\partial}{\partial x_j}\sigma(Q_i)\frac{\partial}{\partial \xi_j}(\sigma(P_i)\hat{\Lambda})\right\}$$
$$=\sigma(H_i)\sigma(P_i)\quad (|\xi|\geq 1)$$

and for $H_i(t)=P_i(t)+\sqrt{-1}Q_i(t)$ $(i=1,\cdots,k)$ *there exists a constant* δ *such that* $|\sigma(H_i-H_j)|\geqq\delta>0$ $(i\neq j)$.

Then, for the operators $J_i=\partial/\partial t+H_i(t)\Lambda$ $(i=1,\cdots,k)$ *we have*

$$\sum_{i+j=\tau\leqq k-1}(nh^{-2})^{(k-\tau)}\int\varphi^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda^j u\right\|^2 dt\leqq C\int\varphi^{-2n}\|J_1\cdot\cdots\cdot J_k u\|^2\,dt \qquad u\in C_0^\infty(\Xi_h) \tag{2.5}$$

for sufficiently small h *and every* n $(\geqq 1)$, *where* $\varphi=(1+t/2h)$ *and* $\Xi_h=\{(t,x);\,-h<t<h\}$.

Proof has been given in [4].

Now set $\mathfrak{D}_{x^0,\varepsilon}=\{x;|x-x^0|<\varepsilon\}$ and $\eta=\varepsilon\nu^{-1/2}\eta^0$ for latice points η^0 in R^ν. Then there exists a partition of the unity such that

$$\Theta_\eta(x)\in C_0^\infty(\mathfrak{D}_{\eta,\varepsilon}),\ \sum_\eta\Theta_\eta^2(x)=1,\ \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\Theta_\eta(x)\right|\leqq C_{\varepsilon,\alpha}. \tag{2.6}$$

Lemma 2 (*S. Mizohata*). *Let* Λ_τ *be an operator defined by* $\widehat{\Lambda_\tau u}(\xi)=\hat{\Lambda}(\xi)\hat{u}(\xi)$ *where* $\hat{\Lambda}_\tau(\xi)(\in C^\infty_{(\xi)})$ *satisfies the conditions*:

$$\hat{\Lambda}_\tau(\xi)=0\ \text{on}\ \{\xi;|\xi|\leqq 1\},\ \left|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\hat{\Lambda}_\tau(\xi)\right|\leqq\gamma_\alpha K(\xi)^{\tau-m|\alpha:\mathfrak{m}|}.$$

Then, for the partition of the unity (2.6) *we have*

$$\sum_\eta\|(\Lambda_\tau\Theta_\eta-\Theta_\eta\Lambda_\tau)u\|^2\leqq C(\gamma_k^2\|u\|^2+\sum_{0<|\alpha|<k}\|(x^\alpha\Lambda_\tau)u\|^2)\quad u\in C_0^\infty(R^\nu)$$

where C *is a constant depending on* ν,ε,τ,k, *and* $M(=\underset{1\leqq j\leqq\nu}{Max}\, m/m_j)$ *and* k *is an integer* $\geqq\tau+(\nu+1)M$.

Proof is essentially the same as that of S. Mizohata [6] if we remark that $|\xi|\leqq CK(\xi)^M(|\xi|\geqq 1)$ and $m|\alpha:\mathfrak{m}|\geqq|\alpha|$.

Lemma 3. *Let* $H_j(t)$ $(j=1,\cdots,k)$ *be operators of* $C_{\mathfrak{m}}^m$ *defined in* (x)*-space with* t *as a parameter.*

Setting $A=\sum_{j=0}^k H_j\Lambda^j\dfrac{\partial^{k-j}}{\partial t^{k-j}}$ $(H_0=1)$ *we assume*

$$\left|\sum_{j=0}^k\sigma(H_j)(t,x,\xi)\hat{\Lambda}(\xi)^j(\sqrt{-1}\,\lambda)^{k-j}\right|^2\geqq\delta^2(\lambda^{2k}+K(\xi)^{2k})\quad(\delta>0,|\xi|\geqq 1). \tag{2.7}$$

Then, for every $\varepsilon_0(>0)$ *we have*

$$(1-\varepsilon_0)\left(\left|\left|\left|\frac{\partial^k}{\partial t^k}u\right|\right|\right|^{2}+|||\Lambda_0^k u|||^2\right)\leqq|||Au|||^2+C_{\varepsilon_0}\sum_{i+j\leqq k-1}\left|\left|\left|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\right|\right|\right|^2 \qquad u\in C_0^\infty(\Xi_h) \tag{2.8}$$

[5]

for sufficiently small h *depending on* ε_0.

Proof. Consider the partition of the unity of (2.6) and let $H_j^{(\eta)}(t)$ be an operator of class $C_{\mathfrak{m}}^m$ with $\sigma(H_j^{(\eta)}(t))=\sigma(H_j)(t,\eta,\xi)$ for each fixed

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5) For a function $u=u(t,x)$, $|||u|||^2$ means $\int|u(t,x)|^2\,dt\,dx$.

η. If we define Λ_j $(j=1,\cdots,k)$ by $\widehat{\Lambda_j u}(\xi)=\hat{\Lambda}(\xi)^j\alpha(\xi)\hat{u}(\xi)$ with $\alpha(\xi)\in C^\infty_{(\xi)}$ $(=0$ for $|\xi|\leqq 1$, $=1$ for $|\xi|\geqq 2)$, we can write

$$|||Au|||^2=\sum_\eta|||\Theta_\eta Au|||^2=\sum_\eta\Big|\Big|\Big|\Theta_\eta\sum_{j=1}^k(H_j(t)-H_j^{(\eta)}(t))\Lambda^j\frac{\partial^{k-j}}{\partial t^{k-j}}u$$

$$+\Theta_\eta\sum_{j=1}^k H_j^{(\eta)}(t)(\Lambda^j-\Lambda_j)\frac{\partial^{k-j}}{\partial t^{k-j}}u+\sum_{j=1}^k(\Theta_\eta H_j^{(\eta)}(t)\Lambda_j-H_j^{(\eta)}(t)\Lambda_j\Theta_\eta)\frac{\partial^{k-j}}{\partial t^{k-j}}u$$

$$+\sum_{j=1}^k(H_j^{(\eta)}(t)-H_j^{(\eta)}(0))\Lambda_j\Theta_\eta\frac{\partial^{k-j}}{\partial t^{k-j}}u+\Big(\frac{\partial^k}{\partial t^k}+\sum_{j=1}^k H_j^{(\eta)}(0)\Lambda_j\frac{\partial^{k-j}}{\partial t^{k-j}}\Big)\Theta_\eta u\Big|\Big|\Big|^2$$

$$\equiv\sum_\eta\Big|\Big|\Big|\sum_{i=1}^5 I_{i,\eta}\Big|\Big|\Big|^2\geqq(1-\varepsilon_1)\sum_\eta|||I_{5,\eta}|||^2-C\varepsilon_1^{-1}\sum_{i=1}^4\sum_\eta|||I_{i,\eta}|||^2\quad(\varepsilon_1>0).$$

Then, we have for $I_{5,\eta}$ by (2.7) and Lemma 2

$$\sum_\eta|||I_{5,\eta}|||^2\geqq(1-\varepsilon_1)\delta^2\Big(\Big|\Big|\Big|\frac{\partial^k}{\partial t^k}u\Big|\Big|\Big|^2+|||\Lambda_0^k u|||^2\Big)-C\varepsilon_1^{-1}\sum_{i+j\leqq k-1}\Big|\Big|\Big|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\Big|\Big|\Big|^2,$$

for $I_{1,\eta}$ and $I_{4,\eta}$

$$\sum_\eta(|||I_{1,\eta}|||^2+|||I_{4,\eta}|||^2)\leqq C(\varepsilon^2+h^2)\sum_{i+j\leqq k}\Big|\Big|\Big|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\Big|\Big|\Big|^2$$

and for $I_{2,\eta}$ and $I_{3,\eta}$ by Lemma 2

$$\sum_\eta(|||I_{2,\eta}|||^2+|||I_{3,\eta}|||^2)\leqq C_\varepsilon\sum_{i+j\leqq k-1}\Big|\Big|\Big|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\Big|\Big|\Big|^2.$$

Since $\sum_{i+j=k}\Big|\Big|\Big|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\Big|\Big|\Big|^2\leqq C\Big(\Big|\Big|\Big|\frac{\partial^k}{\partial t^k}u\Big|\Big|\Big|^2+|||\Lambda_0^k u|||^2\Big)$, we get (2.8) if we fix $\varepsilon_1(>0)$ such that $(1-2\varepsilon_1)^2\geqq(1-\varepsilon_0)$ for given ε_0 and take sufficiently small ε and h depending on ε_1. Q.E.D.

(See References of the following article.)