62. *Use of the Function sin x/x in Gravity Problems*

By C. H. DIX

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The paper of Y. Tomoda and K. Aki,[1] under the above title, describes a method of such simplicity and ease of application as to make it worthy of any small further clarification. Let us look at the "convergence" of the series used. Quotes are used because observed values enter the series as well as analytical expressions, so convergence in the ordinary strict sense may not be applicable.

In their paper, Tomoda and Aki, for clear exposition, take grid points at $\pm n\pi$ with corresponding gravity anomaly values $\Delta g_{\pm n}$ and project downwards to a depth, d in radians. From this downward projection of Δg, a corresponding surface mass density is found, which will yield the same anomaly field as the original one at the datum surface. Then they pass to the more useful case where ξ and δ are the actual horizontal and vertical distances measured in any convenient units and a is the grid spacing between gravity stations measured in the same units. We start from this latter stage of their work.

Let the ratio δ/a be r. Then the mass surface density under the i^{th} grid point will be

$$M(i, r)=\frac{1}{2\pi k^2}\sum_{j=-\infty}^{+\infty}\varphi_j(r)\Delta g_{i+j}$$
$$=\frac{\varphi_0(r)\Delta g_i}{2\pi k^2}+\frac{1}{2\pi k^2}\sum_{j=1}^{\infty}\varphi_j(r)(\Delta g_{i+j}+\Delta g_{i-j}) \tag{1}$$

where (see reference 1), p. 446) k^2 is the gravitational constant and

$$\varphi_j(r)=r\{(-1)^j e^{\pi r}-1\}/\{\pi(j^2+r^2)\}=(-1)^j e^{\pi r}\psi_j(r)-\psi_j(r)$$
$$=\theta_j(r)-\psi_j(r) \tag{2}$$

with

$$\psi_j(r)=r/\{\pi(j^2+r^2)\}. \tag{3}$$

The terms $\psi_i(r)\to r/\pi j^2$ as j increases, for practical values of r, such as $r=1/2$. $\theta_j(r)$ is an alternating sequence of terms of decreasing numerical values.

L. B. W. Jolley[2] (see pp. 22–23) lists summation formulae giving

$$\sum_{j=1}^{\infty}\psi_j(r)=(1/2)\coth\pi r-1/(2\pi r),$$
$$\sum_{j=1}^{\infty}\theta_j(r)=(1/2)(e^{\pi r}/\sinh\pi r)-e^{\pi r}/(2\pi r). \tag{4}$$

So if we use the constant lateral extensions of the Δg values, as Tomoda and Aki[1] (p. 446 bottom lines) do for Vening Meinesz's gravity profile ♯17 in the East Indies, then we can compute $M(i, r)$

in exact closed form. We shall illustrate this using their example below, but let us first obtain the corresponding general formula.

Suppose the sequence of measured gravity stations has indices from 1 to N so that Δg_i is measured if $1 \leq i \leq N$. Let $\Delta g_i = \Delta g_1$ for $i \leq 1$ and $\Delta g_i = \Delta g_N$ for $i \geq N$. In equation (1), for any i, after a certain number of terms $\Delta g_{i+j} + \Delta g_{i-j} = \Delta g_1 + \Delta g_N = C$. Note that C is not dependent on i. For all $\Delta g_{i+j} + \Delta g_{i-j} \neq C$ write $\{(\Delta g_{i+j} + \Delta g_{i-j} - C) + C\}$ in (1). Then the number of non-zero terms with $(\Delta g_{i+j} + \Delta g_{i-j} - C)$ is finite and the terms with C alone can be summed using formulae (4). So (1) becomes

$$M(i, r) = \frac{\varphi_0(r)\Delta g_i}{2\pi k^2} + \frac{1}{2\pi k^2} \sum_{\text{finite}} \varphi_j(r)(\Delta g_{i+j} + \Delta g_{i-j} - C) + \frac{C}{2\pi k^2}\left[\frac{1}{2} - \frac{(e^{\pi r} - 1)}{2\pi r}\right]. \tag{5}$$

Now let us apply this to the Vening Meinesz example. Here $\Delta g_1 = 24$ followed by 33, 28, 28, 58, 110, 70, 120, 320, 355, 360, 350, and $\Delta g_{13} = 320$ mgals. $N = 13$. $\Delta g_1 + \Delta g_{13} = C = 344$ mgals. $\delta = 35$ km., $a = 70$ km.; so $r = \frac{1}{2}$. Remember that the measured values are seldom accurate to three significant figures; that the reduction to datum introduces further uncertainties; that the hypothesis that the volume density depends upon depth only down to a depth, δ, is not quite exact; that our constant extension of Δg's outside the measured region is probably seriously off; and that no real problem is strictly a plane problem. Remembering these natural limitations, we calculate $\varphi_i(1/2)$ only to the third decimal place and this only for a few values of i. Experience in using the method shows that excessive oscillation begins to develop for $r > \frac{1}{2}$. So we are partly concerned with the problem: at what value of j may we safely replace $(j^2 + 1/4)^{-1}$ by j^{-2}? We need the behavior of $(1 + 1/4j^2)^{-1}$. If $(1 + 1/4j^2)^{-1} \approx (1.001)^{-1}$ then $j \approx 16$.

n	1	2	3	4	5	6	7
$\varphi_n(1/2)$	$-.739$	$+.143$	$-.100$	$+.037$	$-.036$	$+.017$	$-.019$
$\Delta g_{7+n} + \Delta g_{7-n} - C$	-114	$+34$	$+39$	$+44$	$+39$	0	0
Products	$+84.3$	$+4.9$	-3.9	$+1.6$	-1.4	0	0

Considering the actual uncertainties and that we wish here only to illustrate the procedure in its best and simplest terms, we tabulate only eight values and include the calculation of the finite sum for $i = 7$ where $\Delta g_7 = 70$. An easy calculation gives $\varphi_0(1/2) = 2.425$. Then excepting the factor $1/2\pi k^2 = 2.38 \times 10^6$ c.g.s., the first term on the right of (5) is $+170 \times 10^{-3}$ c.g.s., the second is $+86 \times 10^{-3}$ c.g.s., (see Table for sum of Products), and the third term is -246×10^{-3} c.g.s., with a total of $+10 \times 10^{-3}$ c.g.s. Multiplying now by 2.38×10^6 c.g.s., we get $+2.38\times 10^4$ g/cm^2. Using 0.6 g/cc as the volume density contrast, the height, $h_7=+0.387$ km. The four neighboring values are $h_5=+1.16$ km., $h_6=+5.98$ km., $h_8=+0.85$ km., and $h_9=+16.8$ km., all computed with a slide rule in 30 minutes. The remaining values are in good agreement with those given by Tomoda and Aki[1] and by C. Tsuboi.[4]

References

1) Y. Tomoda and K. Aki: Proc. Japan Acad., **31**, 443-448 (1955).
2) L. B. W. Jolley: Summation of Series. 2nd ed., Dover, New York (1961).
3) Vening Meinesz: Gravity Expeditions at Sea, II (1934).
4) C. Tsuboi: Bull. Earthq. Res. Inst., **17**, 351-384 (1939).