99. *The Area of Discontinuous Surfaces*

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1. **Introduction.** Let us use the term *rectangle* synonymously with nondegenerate closed interval of the Euclidean plane $\boldsymbol{R}^2$. By a *nonparametric summable surface* on a rectangle we understand a surface of the form $z=F(x, y)$, where F is a summable function defined on the rectangle and assuming finite real values. For brevity, such a surface will often be referred to as an NS surface.

A few authors have already treated the area theory of NS (or more general) surfaces, Cesari [1] and Goffman [3] being representative. The greater part of this paper is concerned with a further contribution to the theory, in which another definition of area will be given to NS surfaces and will be shown equivalent to those of Cesari and Goffman.

We shall apply then our leading idea to *parametric summable surfaces* (§ 6), to obtain a concept of area which, in the special case of parametric continuous surfaces, coincides with the Lebesgue area.

If one seeks to generalize the various results of the existing area theory so as to be valid for parametric summable surfaces, there arise in a natural way a number of research problems. Some of them will be stated toward the end of the paper.

2. **Area of nonparametric summable surfaces.** For any continuous function G on a rectangle I_0, the Lebesgue area of the surface $z=G(x, y)$ will be denoted by $S(G)$ or $S(G; I_0)$, as in [Saks 4]. If G^* is another continuous function on I_0 and E is any nonvoid subset of I_0, the symbol $\rho(G, G^*; E)$ will mean the ordinary distance on E between the two functions, i.e. the supremum of $|G(w)-G^*(w)|$ for $w \in E$. If E is the void set, the same symbol is understood to vanish.

Let $I=[a_1, b_1; a_2, b_2]$ be a rectangle and let h stand for the positive numbers $<2^{-1}\min(b_1-a_1, b_2-a_2)$. We shall write, in the sequel,

$$I_h=[a_1+h, b_1-h; a_2+h, b_2-h].$$

Given on I a finite summable function F, we understand by the *integral mean* of F (for squares of side-length $2h$), the function

$$F_h(x, y)=\frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}F(x+u, y+v)dudv,$$

where the point $\langle x, y\rangle$ ranges over the rectangle I_h. It is well known that F_h is then a continuous function on I_h.

This being said, let now G represent the continuous functions on I. Given F as above, we define the *area* of the nonparametric summable surface $z=F(x,y)$ to be the lower limit of the Lebesgue area $S(G;I_h)$ as $h\to 0$ and $\rho(F_h,G;I_h)\to 0$ simultaneously. This area will be written $L(F)$ or $L(F;I)$. Explicitly, $L(F)$ is the supremum of $L^{(\eta)}(F)$ for all $\eta>0$, where $L^{(\eta)}(F)$ means the infimum of $S(G;I_h)$ for all pairs $\langle h,G\rangle$ such that $h<\eta$ and $\rho(F_h,G;I_h)<\eta$.

REMARK. In view of the definition of the Lebesgue area it is easy to see that we may use, in the above definition of $L(F)$, polyhedral functions on I in place of the continuous functions G.

THEOREM. *If, in particular, the function F is continuous on I, we have the equality $L(F)=S(F)$.*

PROOF. It is obvious that $\rho(F_h,F;I_h)$ tends to zero with h. Hence $L^{(\eta)}(F)\leqq S(F)$ for every $\eta>0$, and so $L(F)\leqq S(F)$.

It remains to derive the converse inequality. We fix in I° (the interior of I) a rectangle J and suppose $\eta>0$ so small that $I_\eta\supset J$. Let $h<\eta$ be arbitrary and let us consider any continuous function G on I such that $\rho(F_h,G;I_h)<\eta$. Noting the inclusion $I_h\supset J$, we have $S(G;J)\leqq S(G;I_h)$, and so the definition of $L(F)$ gives, for fixed η,

$$\inf S(G;J)\leqq L(F), \tag{1}$$

the infimum being taken with respect to all pairs $\langle h,G\rangle$ under consideration.

Now $\rho(F,G;J)\leqq\rho(F,G;I_h)\leqq\rho(F,F_h;I_h)+\eta$, so that the distance $\rho(F,G;J)$ tends to zero with η, uniformly in G. Hence, making $\eta\to 0$ in (1) and using the lower semicontinuity of the Lebesgue area, we obtain in the limit $S(F;J)\leqq L(F)$. Since J is arbitrary, it follows that $S(F)\leqq L(F)$, which completes the proof.

3. **The Goffman area.** If F and F^* are a pair of finite summable functions on a rectangle I, we shall write

$$\delta(F,F^*;E)=(E)\iint|F(x,y)-F^*(x,y)|\,dxdy$$

for any measurable set $E\subset I$. When especially $E=I$, this symbol will often be abbreviated to $\delta(F,F^*)$.

We define the *Goffman area*, $\Phi(F;I)$ or $\Phi(F)$, of an NS surface $z=F(x,y)$ on I to be the lower limit of the Lebesgue area $S(P;I)$ as $\delta(F,P)$ tends to 0, where P is an arbitrary polyhedral function on I. This slight modification of the original definition [Goffman 3] is equivalent to the latter.

If J is a subrectangle of I, the partial function $F|J$ clearly determines an NS surface on J. We write $\Phi(F;J)=\Phi(F|J)$ and find readily that $\Phi(F;J)$ is monotone nondecreasing as a function of J.

Cesari [1], prior to Goffman [3], introduced a definition of the area of a surface $z=F(x,y)$, where F is any finite measurable function on a rectangle. But it was proved in [3] that the two notions of area are equivalent for all NS surfaces. We shall now proceed to show that, for the same class of surfaces, the Goffman area coincides with ours. By the way, it will be seen that our proof of this fact is independent of the two papers just referred to.

4. Lemmas. It is convenient to divide our preliminary considerations into a few lemmas. Throughout the section, F will mean a finite summable function on a rectangle I.

LEMMA (i). *The distance $\delta(F_h, F; I)$ tends to zero with h.*

This may be established as for the proposition (iii) at the bottom of p. 463 of [Cesari 2].

LEMMA (ii). *Suppose that the origin of the plane belongs to the rectangle I and let c denote positive numbers <1. If we define $F^{(c)}(w)=F(cw)$ for $w\in I$, then*

$$\delta(F^{(c)}, F; I)\to 0 \quad as \quad c\to 1.$$

PROOF. We observe first that $F^{(c)}$ as thus defined is a finite summable function on I. To prove the lemma, we shall utilize the same technique as on p. 92 of [Saks 4].

Given any $\varepsilon>0$, there exists a number $\delta>0$ such that, whenever $X\subset I$ is a measurable set with measure $|X|<\delta$,

$$(2) \qquad (X)\iint |F(x, y)|\, dxdy<\varepsilon.$$

This inequality implies that, if X is any such measurable set and if we write $cE=\{cw; w\in E\}$ for sets E in the plane,

$$(3) \qquad (X)\iint |F^{(c)}|\, dxdy=c^{-2}\cdot(cX)\iint |F|\, dxdy<\varepsilon/c^2.$$

On the other hand, by Lusin's theorem, there exists in I° a compact set K on which F is continuous and for which $|I-K|<\delta/2$. Let σ be a number of the open interval $(2^{-1}, 1)$ such that

$$c^{-1}K\subset I \quad \text{and} \quad \rho(F^{(c)}, F; K\cap c^{-1}K)<\varepsilon/|I|$$

whenever $\sigma<c<1$. We then have, for such c,

$$(4) \qquad \delta(F^{(c)}, F; K\cap c^{-1}K)\leqq |I|\cdot(\varepsilon/|I|)=\varepsilon.$$

But evidently $|I-K\cap c^{-1}K|\leqq|I-K|+|I-c^{-1}K|<\delta$, and therefore, by (2) and (3) above, $\delta(F^{(c)}, F; I-K\cap c^{-1}K)<5\varepsilon$ whenever $\sigma<c<1$. If we add this inequality to (4), we obtain $\delta(F^{(c)}, F; I)<6\varepsilon$, and this completes the proof.

LEMMA (iii). *The Goffman area $\Phi(F; I)$ is the supremum of $\Phi(F; J)$ with respect to the rectangles $J\subset I^\circ$.*

PROOF. Without loss of generality we may assume that the origin is in I°. Let $0<c<1$ and consider any polyhedral function P on the rectangle cI. Then $P^{[c]}=c^{-1}P^{(c)}$ is also polyhedral on I and we verify easily $S(P^{[c]}; I)=c^{-2}\cdot S(P; cI)$ as well as

$$\delta(F^{[c]}, P^{[c]}; I)=c^{-3}\cdot\delta(F, P; cI),$$

where again $F^{[c]}=c^{-1}F^{(c)}$. It follows that, for fixed c,

(5) $$c^2\Phi(F^{[c]}; I)\leqq\Phi(F; cI)\leqq\Phi(F; I).$$

On the other hand, lemma (ii) and the two relations

$$\delta(F^{[c]}, F; I)\leqq\delta(F^{[c]}, F^{(c)}; I)+\delta(F^{(c)}, F; I),$$
$$\delta(F^{[c]}, F^{(c)}; I)=(c^{-3}-c^{-2})\cdot(cI)\iint|F|\,dxdy$$

together imply that $\delta(F^{[c]}, F; I)\to 0$ as $c\to 1$. But the Goffman area is clearly lower semicontinuous with respect to the underlying distance δ. Consequently

$$\operatorname*{lower\,lim}_{c\to 1}\Phi(F^{[c]}; I)\geqq\Phi(F; I),$$

and so we deduce from (5) that $\Phi(F; cI)\to\Phi(F; I)$ as $c\to 1$. The assertion follows now directly.

LEMMA (iv). *If G is a continuous function on the rectangle I, we have $S(G_h; I_h)\leqq S(G; I)$ for every $h>0$ considered in* §2.

PROOF. If H means the same functional as introduced on p. 174 of [Saks 4], the formula (7.7) on p. 180 of the same treatise implies that $H(G_{1/2n}; I_{1/2n})\leqq H(G; I)$ for every natural number n such that $(2n)^{-1}$ belongs to the set $\{h\}$. Quite similarly we can obtain the slightly stronger result that $H(G_h; I_h)\leqq H(G; I)$ for every h. On the other hand, by Rado's well-known theorem [Saks 4, p. 179], the functional H coincides with the Lebesgue area for any continuous function on a rectangle. Hence the result.

5. Identification of the Goffman area with ours.

THEOREM. *We have $\Phi(F)=L(F)$ for every finite summable function F on a rectangle I.*

PROOF. We first show that $\Phi(F)\leqq L(F)$. Suppose that J is a rectangle fixed in I° and let P stand for polyhedral functions on I. We confine $h>0$ to so small values that $I_h\supset J$. Then, for any h and any P,

$$\delta(F, P; J)\leqq\delta(F, P; I_h)\leqq\delta(F, F_h; I_h)+\delta(F_h, P; I_h).$$

Since $\delta(F_h, P; I_h)\leqq|I|\cdot\rho(F_h, P; I_h)$, it follows in view of lemma (i) that $\delta(F, P; J)\to 0$ when h and $\rho(F_h, P; I_h)$ tend simultaneously to zero. Therefore, by the definition of $\Phi(F; J)$ and the remark of § 2, we deduce at once $\Phi(F; J)\leqq L(F)$. As J is arbitrary, the desired inequality ensues by lemma (iii).

We have to derive further $L(F)\leqq\Phi(F)$. Let P be as above and let h be arbitrary so long as I_h exists. If $\delta(F, P; I)<4h^3$, then plainly $\delta(F, P; K)/(4h^2)<h$ for all the rectangles $K\subset I$, and therefore $\rho(F_h, P_h; I_h)<h$. But $S(P_h; I_h)\leqq S(P; I)$ by lemma (iv). Consequently, making $h\to 0$, we obtain in the limit $L(F)\leqq\Phi(F)$.

6. Area of parametric summable surfaces. We shall call *parametric summable surface* (or PS *surface*, for short) on a rectangle

I, any map T of I into the ordinary space $\boldsymbol{R}^3$ such that the three coordinate functions of T are summable on I.

The idea that led to our definition of the area of NS surfaces is equally available for PS surfaces and will yield a new kind of surface area. We are interested in this area because it constitutes an extension of the Lebesgue area, as we shall show soon.

Given on a rectangle I a parametric summable surface T, write explicitly $T(w)=\langle X(w), Y(w), Z(w)\rangle$ for $w\in I$, so that X, Y, Z are summable functions on I. We define $T_h=\langle X_h, Y_h, Z_h\rangle$ for each $h>0$ of § 2. This map T_h is clearly a continuous parametric surface on I_h and will be termed *integral mean* of T (with respect to squares of side-length $2h$).

Let R stand for parametric continuous surfaces on I. Given T as in the above, we shall understand by the *area* of T the lower limit of the Lebesgue area $S(R; I_h)$ as $h\to 0$ and $\rho(T_h, R; I_h)\to 0$ simultaneously. (The latter symbol, of course, means the maximum of $|T_h(w)-R(w)|$ on I_h). The new area will be written $L(T)$ or $L(T; I)$.

THEOREM. *We have $L(T)=S(T)$ whenever T is a continuous parametric surface on a rectangle.*

The proof of this is almost the same as for the nonparametric case (see § 2).

THEOREM. *If an* NS *surface $z=F(x, y)$ on a rectangle I is interpreted as a* PS *surface T with the coordinate functions*

$$X(w)=x,\quad Y(w)=y,\quad Z(w)=F(w)\quad (w=\langle x, y\rangle\in I),$$

it holds that $L(T)\leqq L(F)$.

PROOF. It is obvious that $X_h=X, Y_h=Y, Z_h=F_h$ for each $h>0$ of §2, so that $T_h=\langle X, Y, F_h\rangle$. If G is an arbitrary continuous function on I, then $T^*=\langle X, Y, G\rangle$ is a parametric continuous surface on I and fulfils, for every h, the relations

$$\rho(T_h, T^*; I_h)=\rho(F_h, G; I_h)\quad\text{and}\quad S(T^*; I_h)=S(G; I_h),$$

the latter being a well-known property of the Lebesgue area. It follows that $L(T)$ does not exceed the lower limit of $S(G; I_h)$ as $h\to 0$ and $\rho(F_h, G; I_h)\to 0$. In other words, we have proved $L(T)\leqq L(F)$.

7. **Area of planar summable maps.** A map T of a rectangle I into the plane $\boldsymbol{R}^2$ will be called *summable* if both of its coordinate functions are summable over I. We can define the *area*, $L(T)$ or $L(T; I)$, of such a map in the same way as for PS surfaces. The details are left to the reader.

8. **Problems.** (i) Investigate whether the equality $L(T)=L(F)$ holds in the last theorem of § 6.

(ii) Generalize our definition of the area of NS surfaces [or PS surfaces] so that the new area exists for all nonparametric [or parametric] *measurable* surfaces on a rectangle. Examine whether, in the nonparametric case, the Cesari area (see § 3) always coincides with the new one.

(iii) *Is the area of* PS *surfaces invariant under Lebesgue equivalence* [Cesari 2, p. 62], like the Lebesgue area of continuous parametric surfaces? Explicitly, let θ be a homeomorphism of a rectangle I into another rectangle I^*. If T and T^* are PS surfaces on I and I^*, respectively, such that $T(w)=T^*(\theta w)$ for all $w \in I$, does it follow that $L(T)=L(T^*)$?

(iv) *Extend, if possible, the Kolmogorov principle for the Lebesgue area* [Cesari 2, p. 53] *to the area of* PS *surfaces.* Explicitly, let τ be a continuous map of the space $\boldsymbol{R}^3$ into itself such that

$$|\tau(p)-\tau(p')| \leqq |p-p'| \quad \text{whenever} \quad p, p' \in \boldsymbol{R}^3.$$

If T is any PS surface on a rectangle I, the composite map $T^*=\tau T$ is likewise a PS surface on I. In fact, T^* is a measurable map, and if w_0 denotes the centre of I, we have

$$|T^*(w)-T^*(w_0)| \leqq |T(w)-T(w_0)|$$

for all $w \in I$. So that $|T^*(w)-T^*(w_0)|$, and hence the surface T^* itself, must be summable on I. Now the problem requires us to examine for validity the inequality $L(T^*) \leqq L(T)$.

(v) Let $T=\langle X, Y, Z\rangle$ be a PS surface on a rectangle, and consider the three planar maps $T^1=\langle Y, Z\rangle$, $T^2=\langle Z, X\rangle$, $T^3=\langle X, Y\rangle$ which are clearly summable (see § 7). Is it always true that

$$L(T) \leqq L(T^1)+L(T^2)+L(T^3)\,?$$

(vi) The two notions *bounded variation* and *absolute continuity* are very important in area theory, as is well known. Generalize these notions so as to become available for PS surfaces on a rectangle. What results will then hold in place of the First and Second Theorems of [Cesari 2, § 1]?

REMARK. Needless to say, problem (v) is closely related to part of problem (vi).

References

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