# 156. The Role of Mollifiers in S Matrix Theory

By Hideo YAMAGATA

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§ 1. **Introduction.** In order to describe $S$ matrix in the form $S=T\left(\exp i\int g(x)L(x)dx\right)=\sum_{n=0}^{\infty}(i^n/n!)\int T(L(x_1)\cdot L(x_2)\cdots L(x_n))g(x_1)g(x_2)\cdots g(x_n)dx_1\cdots dx_n$, a function $g(x)$ is used. By using the discussions in [4–6], it can be shown that this function $g(x)$ does not necessarily play the role of testing functions but mollifiers. Namely the direct product of the same $g(x)$ contained in $(D)$ cannot construct the dense set in $(D)\otimes(D)\otimes\cdots\otimes(D)$, where $(D)$ is the space consisting of $C^\infty$ functions with compact carrier defined by L. Schwartz [2]. Even in infinite direct product space constructed by $(D)$ the same problem happens. From this it is obvious that the above description of $S$ matrix is very imcomplete. In § 2 we will show this. This result necessarily shows the incompleteness of the description of causality condition, too. Namely our causality condition is effective to only the $S$ matrix described by the form $S(g)$. Furthermore, it is the limit of formulas showing a sort of causality condition which is effective to non local Lagrangian. To describe the causality condition directly, we must use the element of ranked space instead of $g(x)$ [7–8]. We will show these facts in § 3.

§ 2. **The product of distributions in $S$ matrix theory.** Afterward, we use the following notations.

Let $T(u(x)u(y))$ denote the product

$$T(u(x)u(y))=\begin{cases} u(x)u(y) & \text{for } x^0>y^0 \\ \pm u(y)u(x) & \text{for } x^0<y^0. \end{cases}$$

(For Bose operators, the sign $+$ is used, and for Fermi operators the sign $-$ is used.) This product is called chronological product or $T$-product.

Let $T(u(x)\otimes u(y))$ denote the direct product

$$T(u(x)\otimes u(y))=\begin{cases} u(x)\otimes u(y) & \text{for } x^0>y^0 \\ \pm u(y)\otimes u(x) & \text{for } x^0<y^0. \end{cases}$$

(For Bose operators the sign $+$ is used, and for Fermi operators the sign $-$ is used.) This direct product is called chronological direct product or $T$ direct product.

Let $(D)$ denote the space of $C^\infty$ functions with compact carrier which has the topology defined by L. Schwartz in [2], $\prod_{i=1}^{\infty}\otimes(D)$ denote the infinite direct product of $(D)$, and $D_\infty$ denote the closure of the linear aggregate of the elements in $\prod_{i=1}^{\infty}\otimes(D)$ by means of

Tychonoff's weak topology with respect to infinite direct product. Since the concept of $T$ direct product is used in the infinite direct product of the Lagrangians $L(x_i)(i=1, 2, \cdots)$, the concept of $T$ direct product need not use in the infinite direct product of testing functions.

Let $D_1$ denote the closure of the linear aggregate of $\{\prod_{i=1}^{\infty}\otimes g(x)\}$ (for the same $g(x)\in(D)(E^4)$) by the topology in $D_\infty$. Here the description of $S$ matrix by using $D_\infty$ is also incomplete. But to show the incompleteness of the description $S(g)$, it does not matter to use $D_1$ contained in $D_\infty$.

**Lemma 1.** *$D_1$ is not contained in $D_\infty$ densely.*

**Proof.** Let's show that the element $g_1\otimes g_2\otimes g_1\otimes\cdots(g_1\neq g_2)$ in $D_\infty$ is not contained in $\bar{D}_1$ obtained by the topology in $D_\infty$. Namely, choose the neighbourhoods $V_i=g_i+V(\{\varepsilon_{ij}\}, \{\Omega_{ij}\}, \{k_{ij}\})$ $(0<\varepsilon_{ij}<\max|g_1-g_2|/2)$ for $1\leqq i\leqq 3$ and $V_i=(D)$ for $i\geqq 4$, and construct the neighbourhood $\prod_{i=1}^{\infty}\otimes V_i$ in $D_\infty$. Here, $g_i$ is the $i$ th component of $g_1\otimes g_2\otimes g_1\otimes\cdots$ and $V(\{\varepsilon_{ij}\}, \{\Omega_{ij}\}, \{k_{ij}\})$ is the neighbourhood in $(D)$ defined by L. Schwartz [2]. Then $(\prod_{i=1}^{\infty}\otimes V_i)\cap D_1=\phi$. Hence, this lemma is proved.

**Lemma 2.** *$D_\infty$ is a locally convex linear topological space.*

**Proof.** From its construction it is obvious that the space $D_\infty$ is a linear topological space. Hence, it is sufficient to show that the topology in $D_\infty$ is locally convex. The family of the neighbourhood in $\prod_{i=1}^{\infty}\otimes(D)$ is $\{\prod_{i=1}^{\infty}\otimes V_i\}$ where $V_i=(D)$ except for finite $i$. Since $(D)$ is a local convex space, we can take the family of convex neighbourhoods $V_i$ which define the same topology in $(D)$. Hence the topology in $D_\infty$ can be defined by the convex neighbourhoods $\prod_{i=1}^{\infty}\otimes V_i$. Then this lemma is proved.

Our purpose is to clarify the exact meaning of the calculas used in the usual quantum field theory. In quantum field theory the deeper consideration is not yet adopted to using cut off process, taking various limits, using mollifiers and using testing functions, etc. In the case to take limit after the cut off process, we feel as if it is right and exact. But even in this case the difference between the use of the mollifier and the use of the testing functions appears. Namely, in $S(g)$, the use of the mollifier is still effective. In the following we show this.

Let $E$ denote a locally convex linear topological space consisting of testing functions and $F(\neq E)$ denote a closed subset of $E$ with induced topology.

**Theorem 1.** *Each element of $F'$ corresponds to a class of $E'$.*

**Proof.** If the domain of the functional element in $E'$ is restricted to $F$, then this restricted element is always contained in $F'$. On the other hand, the functional element in $F'$ can be always extended to the functional elements in $E'$, by using Hahn-Banach's extension theorem [3]. Then this theorem is proved.

Suppose that the set of the sequences (of the elements in $(D)$) $\Psi=[\{\psi_n\}]$ has the property $\lim_{n\to\infty}\psi_n=1$ in $D'_\infty$.

**Theorem 2.** *If $\Phi$ is the subset of $\Psi$, then $\lim_{n\to\infty} T\varphi_n(\{\varphi_n\}\in\Phi)$ is not so ambiguous as $\lim_{n\to\infty} T\psi_n(\{\psi_n\}\in\Psi)$.*

This theorem is very important. But it is evident. Then we omit the proof of this theorem. From Lemmas 1–2, Theorems 2–3 can be adopted to $D_\infty$ and $D_1$, and it follows that our $S$ matrix theory cannot fully describe the character of $S(1)=T\left(\exp i\int L(x)dx\right)$ even in the limit, because $g(x)$ does not play the role of testing functions but mollifiers [6]. Namely from the above arguments we can see the justice of the selection of the suitable conditional convergence sequence in the multiplication of distributions.

§ 3. **Causality condition.** Here we use the two domains $G_1$, $G_2$ having the relation such that all points of one of them $(G_1)$ lie in the past with respect to certain time instant $t$, while all points of the other sub-region $(G_2)$ lie in the future with respect to $t$. We denote this relation by $G_2>G_1$. Suppose that the function $g(x)$ is represented as a sum of two functions $g(x)=g_1(x)+g_2(x)$ one of which $(g_1)$ differs from zero only in $G_1$, while the second $(g_2)$ differs from zero only in $G_2$.

**Definition 1.** The causality condition is one such that $S(g_1+g_2)=S(g_2)S(g_1)$ for $G_2>G_1$.

In this definition non local Lagrangians constructed by using the mollifier $g(x)$ can be seen. $S$ matrix constructed by using this non local Lagrangian satisfies a sort of causality condition for $g_1$ and $g_2$ whose carriers have the sufficiently large distance. Using this non local causality, the local causality is defined. For the direct definition of local causality we must use the element in ranked space. In this paragraph one of our purpose is to obtain the necessary and sufficient condition of causality in Definition 1. Suppose that using the two functions $g'_2(x)$ and $g''_2(x)$ which is different from zero only in $G_2$, $g'(x)$ and $g''(x)$ are constructed by the sums $g'(x)=g_1(x)+g'_2(x)$ $g''(x)=g_1(x)+g''_2(x)$. Here $g_1$ is a function defined in the beginning of this paragraph.

**Lemma 3.** *If and only if $S(g_1+g_2)=S(g_2)S(g_1)$ for any $g_1$ and $g_2$, then $S(g'')S^\dagger(g')=S(g''_2)S^\dagger(g'_2)$ for any $g_1$, $g'_2$, and $g''_2$.*

**Proof.** From $S(g_1+g_2)=S(g_2)S(g_1)$, it follows that $S^\dagger(g_1+g_2)=S^\dagger(g_1)S^\dagger(g_2)$.

Hence, we obtain the relation

$$S(g'')S^\dagger(g')=S(g''_2)S(g_1)S^\dagger(g_1)S^\dagger(g'_2)=S(g''_2)S^\dagger(g'_2).$$

If the relation $S(g_1+g_2)\neq S(g_2)S(g_1)$ holds good for some pair of above functions $g_1$ and $g_2$, then the relation

$$S(g_1+g_2)S^\dagger(g_1+0)=S(g_1+g_2)S^\dagger(g_1)\cdot 1\neq S(g_2)S(g_1)S^\dagger(g_1)\cdot 1=S(g_2)S^\dagger(0)$$

is obtained. Consequently, from $S(g'')S^\dagger(g')=S(g_2'')S^\dagger(g_2')$, the relation $S(g_1+g_2)=S(g_2)S(g_1)$ is deduced.

If $g'(y)$ and $g''(y)$ are defined by the relations $g'(y)=g(y)$ and $g''(y)=g(y)+\delta g(y)$ for an infinitesimal variation $\delta g(y)$ which is different from zero only for $y^0>t$, then $S(g'')=S(g)+\delta S(g)$ where $\delta S(g)=\int_{y_0>t}\sum_{n=1}^{\infty}(1/n!)(\delta^n S/\delta g(y)^n)\delta g(y)^n dy$. Then $S(g'')S^\dagger(g')=S(g)S^\dagger(g)+\delta S(g)S^\dagger(g)=1+\delta S(g)S^\dagger(g)$, and according to $S(g'')S^\dagger(g')=S(g_2'')S^\dagger(g_2')$ it does not depend on the variation of the function $\delta g(x)$ such that $x^0<t<y^0$. Consequently $\delta S(g)S^\dagger(g)=\int_{y_0>t}\sum_{n=1}^{\infty}(1/n!)\delta^n S/\delta g(y)^n\,\delta g(y)^n S^\dagger(g)dy$ does not depend on the variation $\delta g(x)$. Hence we obtain the following

**Lemma 4.** *If and only if $S(g'')S^\dagger(g')=S(g_2'')S^\dagger(g_2')$ for any functions $g_1$, $g_2'$, and $g_2''$ (the same as one used in Lemma 1), then $\delta/\delta g(x)\{(\delta S(g)/\delta g(y))S^\dagger(g)\}=0$ is obtained for $\delta g(x)$ and $\delta g(y)$ which have the property $x\lesssim y$ (for these carriers). Here $x\sim y$ means that the points $x$ and $y$ are separated by a spacelike interval.*

The necessity is easily seen by the above arguments. But the sufficiency is not so evident. The necessary and sufficient condition for $S(g'')S^\dagger(g')=S(g_2'')S^\dagger(g_2')$ can be represented by the most usual form $\int_{y_0>t}\delta/\delta g(x)\,(\delta^n S/\delta g(y)^n\,S^\dagger(g))\delta g(y)^n dy=0$ for any $n$. The equivalence between this and the relation $\delta/\delta g(x)(\delta S(g)/\delta g(y)\,S^\dagger(g))=0$ (for $t<y^0$) is shown in the form of the following Lemma 5 which can be proved easily.

**Lemma 5.** *If $\delta/\delta g(x)(\delta S(g)/\delta g(y)S^\dagger(g))=0$ (for $x^0<t<y^0$) is satisfied for any $g$, then*

$$\int_{y_0>t}\delta/\delta g(x)\,\{(\delta g(y)\,\delta/\delta g(y))^n\,S\cdot S^\dagger(g)\}dy=0\quad(n=1,2,\cdots)$$

*is satisfied for any $\delta g(x)$, $\delta g(y)$ such that $x^0<t<y^0$.*

The inverse of this lemma is easily obtained by taking the various $\delta g(y)$ as testing functions. Comparing with the ordinary Taylor expansion, the following lemma is deduced.

**Lemma 6.** *If $\int_{y_0>t}\delta/\delta g(x)\{(\delta g(y)\,\delta/\delta g(y))^n\,S\cdot S^\dagger(g)\}dy=0$ holds good for any $\delta g(x)\,\delta g(y)$ such that $x^0<t<y^0$, then*

$$\int_{y_0>t}\delta/\delta g(x)\{(\delta g_1(y)\,\delta/\delta g_1(y))\cdots(\delta g_n(y)\,\delta/\delta g_n(y))\,S\cdot S^\dagger(g)\}dy=0$$

*holds good for any different variations $\delta g(x)$, $\delta g_1(y)$, $\delta g_2(y),\cdots,\delta g_n(y)$ such that $x^0<t<y^0$, where $n$ is a fixed natural number.*

According to Lemmas 3–5, we obtain the following

**Theorem 3.** *If the relation* $\delta/\delta g(x)\,(\delta S(g)/\delta g(y)\,S^{\dagger}(g))=0$ *(for* $x^0<t<y^0$*) holds good for any* $g$*, then only the relation* $S(g'')S^{\dagger}(g')=S(g_2'')S^{\dagger}(g_2')$ *holds good for any* $g_1$*,* $g_2'$*,* $g_2''$ *used in Lemma* 1. *Namely it follows that only the relation* $S(g_1+g_2)=S(g_2)\,S(g_1)$ *holds good for* $g_1$*,* $g_2$ *defined in the beginning of this paragragh.*

As it is seen in the argument of the beginning of this paragraph, this causality condition is defined by a sort of non-local causality. To define the local causality directly, let's use the following elements of ranked space consisting of the elements in $(D)$ [7]. Namely, $V(F, \nu; 0)$ is the set of the function $g(x)=p(x)+r(x)$ ($p(x)\in(D)$, $r(x)\in(D)$) satisfied the conditions

$$\text{(A)}\quad r(x)=0 \text{ in } F \qquad \text{(B)}\quad \int_0^1 |p(x)|dx<2^{-\nu} \qquad \text{(C)}\quad \left|\int_0^1 r(x)dx\right|<2^{-\nu}$$

where $F\subset[0, 1]$. And $V(F, \nu; f)=\{g(x);\ g(x)-f(x)\in V(F, \nu; 0),\ f(x)\in(D)\}$. Suppose that we can select the sequence of neighbourhoods $\{V(F_i, \nu_i; f_i)\}$ such that

(a) $V(F_1, \nu_1; f_1)\supseteq V(F_2, \nu_2; f_2)\supseteq\cdots$,

(b) $f(x)=\lim\limits_{m\to\infty} f_m(x)$

(c) $f_{2n}=f_{2n+1}$, $\nu_{2n}<\nu_{2n+1}$.

Now let's investigate the various meaning of $T\{f_n\}$ for $T\in(D')$. In the definition of $(D')$, the topology in $(D)$ by L. Schwartz different from the above $V(F, \nu; f)$ is used. About these two topologies, the various pairs are considerable. Here we do not comment on this.

In [8], Бонди construct the $A$-integral representation of $T\in(D')$. Namely, using an $A$-integrable function $F$, $T$ is represented by

$$T\varphi=(A)\int F\varphi d\varphi=\lim_{n\to\infty}\int[F\varphi]_n dx=\lim_{n\to\infty}\int[F]_n\varphi dx \quad \text{for } \varphi\in(D).$$

Here $[F]_n=\begin{cases} F & \text{for } x \text{ such that } |F|\leqq n \\ 0 & \text{for } x \text{ such that } |F|>n, \end{cases}$ and $\operatorname{mes}\{x;\ |F|>n\}=0(1/n)$.

By the selection's method of the pair of sequences $(m(l), n(l))$ such that $\lim\limits_{l\to\infty} m(l)=\infty$ and $\lim\limits_{l\to\infty} n(l)=\infty$, $\lim\limits_{l\to\infty}\int[F]_n(x)f_m(x)dx$ can take different values. To define the local causality condition directly this character must be used. Namely, by using $\{f_m^{(i)}\}$ ($f_m^{(i)}\in(D)(E^4)$ and $\lim\limits_{m\to\infty} f_m^{(i)}=\delta$), local causality condition can be defined by the relation $S(\{f_m^{(1)}+f_m^{(2)}\})=S(\{f_m^{(1)}\})S(\{f_m^{(2)}\})$ (for any considerable $m(l)$, $n(l)$) directly. If the carrier of the element of ranked space $g_1$ and $g_2$ are one point, the local causality condition is defined directly in a sense by using these $g_1$ and $g_2$. But even this local causality condition can be adopted to only the restricted description of $S$ matrix $S(g)$. Namely even Lemma 6 cannot be effectively adopted except for $S(g)$.

## References


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