176. An Approach to Locally Convex Topological Linear Spaces*)

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In this note, I shall explain some ideas to study locally convex topological linear spaces and locally m-convex topological algebras.[1] The detailed results will be published in future.

An inconvenient point to discuss a locally convex topological linear space E is that, in general, there exists a fundamental family of infinitely many semi-norms $\{p_\alpha\}$ on E to define the topology. To eliminate such an inconvenient point and to develop a new theory, consider the mapping $x \to \|x\| = \{\cdots, p_\alpha(x), \cdots\}$ with the weak topology, then we have a continuous mapping $x \to \|x\| \in \prod_\alpha R^1$, where R^1 is the set of real numbers. The order in $\prod_\alpha R^1$ is defined by the coordinate-wise order.

Therefore, we have a kind of norm $\|x\|$ which takes on values in $\prod_\alpha R^1$, and $\|x\|=0$ implies $x=0$, since $\{p_\alpha\}$ is fundamental. Hence we have the following proposition.

For any locally convex topological linear space E, there is a norm $\|x\|$ which takes on values in a product space of the real line. The norm satisfies the following conditions:

1) $\|x\| \geq 0$, $\|x\|=0$ if and only if $x=0$,
2) $\|x+y\| \leq \|x\| + \|y\|$,
3) $\|\lambda x\| = |\lambda| \, \|x\|$, where λ is a scalar.

Further, any locally m-convex topological algebra (in the sense of E. A. Michael [2]) admits a norm satisfying the conditions 1), 2), 3) and $\|xy\| \leq \|x\| \, \|y\|$. For a non-Archimedean topological linear spaces by A. F. Monna [4], we also have a similar result: Let E be a non-Archimedean topological linear space (in the sense of A. F. Monna), then there is a norm $\|x\|$ taking on values in a product space of the real line satisfying

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1) I introduced these considerations, when I gave my lecture course (the 2nd semester of 1963) on topological linear spaces at the Universidad del Sur, Bahia Blanca, Argentina. Further, I spoke of these topics and some results in my lectures at Montevideo, Sao Paul, and Refice Universities, and the IMPA at Rio de Janeiro. On the other hand, by the kind suggestion of Professor Yu. Smirnov, I knew a similar discussion by Tashkent group (see [6]), though their articles are not accessible to me. For a brief summary, see Yu. Smirnov [5].

4) $\|x\| \geq 0$, $\|x\|=0$ if and only if $x=0$,

5) $\|x+y\| \leq \text{Max}\ (\|x\|, \|y\|)$,

6) $\|\lambda x\|=|\lambda|\,\|x\|$, where λ is any element of the coefficient field K.

A general remark is that convergences on E are not necessary sequential. For a metrizable space E, convergences are defined by ordinary sequences. By the concept mentioned above, we can omit the assumption of separability in many theorems, as Tashkent group had already remark to make. The product space $\prod_{\alpha} R^1$ is closed for coordinatewise addition and multiplication, and is a complete topological algebra for the weak topology. On the other hand, as an order is defined on the product space, it is considered as a vector lattice, or rather a topological semiring (in the sense of the present author) or ring with an order.

The purpose of the author is to apply the consideration above to the study of locally convex topological linear spaces. Under the consideration of the R-norm defined in above, we can treat many results, especially some results on the projective tensor product in the simple form. We can develop the theory by Schatten method [3].

The integrations of a function having the values in an abstract space (for example, topological linear space valued functions) were considered by several mathematicians. We can also develop an abstract integration theory based on the R-norm on a locally convex topological linear space.

On the other hand, these considerations are led to a generalizations of R-norm. Let E be an abstract set, R the product space of the set of real numbers. We introduce an order and a topology in the sense mentioned above in R. We define an R-metric on E which takes on values in R as follows:

7) $\rho(x, y) \geq 0$, where the equality holds if and only if $x=y$,

8) $\rho(x, y)=\rho(y, x)$,

9) $\rho(x, y)+\rho(y, z) \geq \rho(x, y)$.

An abstract metric without topology has been studied by some mathematicians (see Z. P. Mamuzić [1], pp. 121–150).

As an example, we shall define a Hilbert space. A pre-Hilbert space E with an R-norm (R-inner product) (x, y), $x, y \in E$ (where R is considered as the product space of complex numbers) is defined as

10) $(x, x) \geq 0$, $(x, y)=0$ if and only if $x=y$,

11) $(x, y)=\overline{(y, x)}$,

12) $(\alpha x, y)=\alpha(x, y)$,

13) $(x_1+x_2, y)=(x_1, y)+(x_2, y)$.

Then E is a generalization of a Hilbert space with countably inner products $(x, y)_n$, $n=1, 2, \cdots$.

We obtain characterizations of pre-Hilbert spaces. As an example, by the well known argument, we have P. Jordan-J. von Neumann characterization.

References

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