141. Some Mapping Theorems for the Numerical Range

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The purpose of the present note is to prove some mapping theorems for the numerical range of a linear operator, somewhat analogous to the spectral mapping theorem. Because of the peculiarity that the numerical range is always convex, the theorems are rather restricted in validity compared with the spectral mapping theorem.

In what follows we mean by an operator A a bounded linear operator in a Hilbert space H with domain H. The numerical range and the spectrum of A are denoted by $W(A)$ and $S(A)$, respectively. It is well known that $S(A)\subset\overline{W(A)}$ ($\overline{\ \ }$ denotes the closure) and that $\overline{W(A)}$ is the closed convex hull of $S(A)$ if A is normal.

Also we need the notion of the *convex kernel* K of a non-empty set E in the complex plane; K is the set of all points z such that E is star-shaped relative to z. It is known[1] that K is a convex subset of E, $K=E$ if E is convex, and that K is compact if E is.

Theorem 1. Let $f(z)$ be a rational function with $f(\infty)=\infty$. Let E' be a compact convex set in the complex plane, let $E=f^{-1}(E')$ and let K be the convex kernel of E. If A is an operator with $W(A)\subset K$, then $W(f(A))\subset E'$.

Remark 2. Under the assumptions of the theorem, E', E, and K are all compact and f has no poles in E. Since $S(A)\subset\overline{W(A)}\subset K\subset E$, $f(A)$ is well defined. K may be empty, in which case the theorem is of no use. For K to be non-empty, it is necessary that E be connected and contain all critical points of f (so that E' contain all branch points of the inverse function f^{-1}).

Corollary 3.[2] If $W(A)$ is a subset of the closed unit disk, the same is true of $W(A^n)$, $n=2, 3, \cdots$.

For the proof of Theorem 1 and other theorems given below, we use the following lemma, the proof of which is trivial. We set $Re\,A=(A+A^*)/2$, $Im\,A=(A-A^*)/2i$, and note that $([Re\,A]u, u)=Re(Au, u)$, $([Im\,A]u, u)=Im(Au, u)$ for any $u\in H$.

Lemma 4. Let A be a nonsingular operator. Then $Re\,A\geqq 0$ is

1) See [2] and [5].

2) This theorem is due to C. A. Berger [1]. The author was told that it was also proved by C. M. Pearcy. For $n=2^m$ it had been proved earlier by H. Fujita (unpublished).

equivalent to $Re\,A^{-1}\geqq 0$, and $Im\,A\geqq 0$ is equivalent to $Im\,A^{-1}\leqq 0$.

Proof of Theorem 1. If K is empty, there is nothing to be proved. If K is contained in a straight line, the same is true of $W(A)$ so that A is normal. Then $f(A)$ is also normal and, since $S(f(A))=f(S(A))\subset f(\overline{W(A)})\subset f(K)\subset E'$ by the spectral mapping theorem, we have $W(f(A))\subset E'$.

Thus we may assume that the convex set K is the closure of its interior K°. We may further assume that $\overline{W(A)}\subset K^{\circ}$; if we prove the theorem in this special case, the general case can be dealt with by considering λA, $0<\lambda<1$, and going to the limit $\lambda\to 1$ (assuming $0\in K^{\circ}$ without loss of generality).

Thus we shall assume $K=\overline{K^{\circ}}$ and $\overline{W(A)}\subset K^{\circ}$ in the remainder of the proof. Then $S(A)\subset\overline{W(A)}\subset K^{\circ}$ and $S(f(A))=f(S(A))$ is in the interior of E'. It follows that $(f(A)-c')^{-1}$ exists as a bounded operator for any c' on the boundary C' of E'.

To prove the theorem it suffices to show that for any supporting line l' of E', $W(f(A))$ is contained in the closed half-plane bounded by l' and containing E'. Here we may restrict l' to genuine tangents to C', for the tangents exist and change continuously except possibly at a countable number of points. Thus we have only to prove that $Im\,e^{-i\theta'}((f(A)-c')u,u)\geqq 0$ for all $u\in H$ or, equivalently,

$$Im\,[e^{-i\theta'}(f(A)-c')]\geqq 0 \tag{1}$$

for each $c'\in C'$ where the tangent l' exists, where θ' is the angle of inclination of l'(with respect to the positive real axis) oriented in such a way that E' lies to the left of l'. In virtue of Lemma 4, (1) is equivalent to

$$Im\,[e^{i\theta'}(f(A)-c')^{-1}]\leqq 0. \tag{2}$$

We may further assume that c' is not a branch point of f^{-1}, for there are only finitely many branch points. Then the poles $c_1,\cdots,c_n$ of $(f(z)-c')^{-1}$ are all simple, with $f'(c_k)\neq 0$, and we have the partial fraction expansion (note that $f(\infty)=\infty$)

$$(f(z)-c')^{-1}=\sum_k f'(c_k)^{-1}(z-c_k)^{-1}. \tag{3}$$

We have the same identity with z replaced by A. To prove (2), therefore, it suffices to show that for each k

$$Im\,[e^{i\theta'}f'(c_k)^{-1}(A-c_k)^{-1}]\leqq 0. \tag{4}$$

By Lemma 4, (4) is in turn equivalent to

$$Im\,[e^{-i\theta'}f'(c_k)\,(A-c_k)]\geqq 0. \tag{5}$$

Since $f(c_k)=c'$ and $f'(c_k)\neq 0$, f is conformal on a neighborhood of c_k to a neighborhood of c'. Since $c'\in C'$, c_k is on the boundary C of E and C has a tangent l_k at c_k. Let θ_k be the angle of inclination of l_k. Then arg $f'(c_k)=\theta'-\theta_k$ so that $e^{-i\theta'}f'(c_k)=e^{-i\theta_k}\,|f'(c_k)|$. But

$Im\,[e^{-i\theta_k}(A-c_k)]\geqq 0$, for $W(A)\subset K$ is in the half-plane to the *left* of l_k, as is easily seen by a simple geometric consideration. This proves (5) and completes the proof of Theorem 1.

Theorem 1 is "global" in the sense that the behavior of f in the whole plane is essential; if f were defined only on a subset D of the plane, the theorem is not applicable. We shall now prove some mapping theorems which are "local" in this respect, but these are restricted to functions defined on special domains: disks and half-planes. We shall state the theorems for the closed *unit disk* $D: |z|\leq 1$ and the closed *right half-plane* $P: Re\, z\geqq 0$, but it is obvious how they should be modified in the general cases.

Theorem 5.[3] Let $f(z)$ be holomorphic on D and map D into D, with $f(0)=0$. If $W(A)\subset D$, then $W(f(A))\subset D$.

Theorem 6. Let $f(z)$ be holomorphic on D and map D into P. If $W(A)\subset D$, then $W(f(A))\subset P-Re\,f(0)$ (the half-plane $Re\, z\geqq -Re\; f(0)$).

Theorem 7. Let $f(z)$ be holomorphic on P. If $W(A)\subset P$, then $W(f(A))$ is a subset of the closed convex hull of $f(P)$.

Remark 8. These three theorems are similar in some respects but are essentially different. The reason for difference is that the linear transformations ϕ that map D onto P (for example $\phi(z)=(1-z)/(1+z)$) do not have the same effect on the numerical range. In other words, the condition $W(A)\subset D$ is weaker than $W(\phi(A))\subset P$; the latter is equivalent to $\|A\|\leqq 1$ rather than to $W(A)\subset D$. For this reason, the assumption of Theorem 7 is rather strong and the conclusion is strong accordingly. For the same reason, the term $-Re\; f(0)$ in Theorem 6 cannot be dropped. In view of these facts, it is rather remarkable that Theorem 5 is true.

Proof of Theorem 6. We have the following well-known formula[4] representing $f(z)$ in terms of the real part of its boundary values:

$$f(z)=i\,Im\,f(0)+\frac{1}{2\pi}\int_0^{2\pi}[Re\,f(e^{it})]\frac{e^{it}+z}{e^{it}-z}dt$$
$$=-\overline{f(0)}+\frac{1}{\pi}\int_0^{2\pi}[Re\,f(e^{it})]\,(1-e^{-it}z)^{-1}dt.$$

The same formula is true when z is replaced by A, so that

$$Re\,f(A)=-Re\,f(0)+\frac{1}{\pi}\int_0^{2\pi}[Re\,f(e^{it})]\,[Re\,(1-e^{-it}A)^{-1}]dt.$$

But $W(A)\subset D$ implies $Re\,(1-e^{-it}A)\geqq 0$ and hence $Re[(1-e^{-it}A)^{-1}]\geqq 0$

3) Corollary 3 is also a consequence of this theorem, which is due to Sz. Nagy (the author owes this and other informations to P. D. Lax).

4) See [4], p. 570.

by Lemma 4. Since $Re\, f(e^{it}) \geqq 0$ by hypothesis, we have $Re\, f(A) \geqq -Re\, f(0)$.

Proof of Theorem 5. Set $g(z)=(1+af(z))/(1-af(z))$, where a is a constant with $|a|<1$. Since g maps D into P with $g(0)=1$, an application of Theorem 6 to g shows that $W(g(A)) \subset P-1$ or $Re(g(A)+1) \geqq 0$. Since $g(A)+1=2(1-af(A))^{-1}$, it follows by Lemma 4 that $Re(1-af(A)) \geqq 0$. Since a is arbitrary as long as $|a|<1$, we have $W(f(A)) \subset D$.

Proof of Theorem 7. Set $B=(A-1)(A+1)^{-1}$; then $\|B\| \leqq 1$ (see Remark 8). We have $f(A)=h(B)$, where $h(z)=f((1+z)/(1-z))$. It is obvious that $f(P) \subset h(D)$.

h is not necessarily holomorphic on the closed set D, but $h_\lambda(z)=h(\lambda z)$ is for each positive $\lambda<1$. To prove the theorem, it suffices to show that $W(h_\lambda(B))$ is a subset of the closed convex hull of $h(D)$. To this end it suffices to show that $W(h_\lambda(B))$ is contained in any closed half-plane Π that contains $h(D)$. Let p be an interior point of Π and let p' be its image with respect to the boundary of Π. Then $W(h_\lambda(B)) \subset \Pi$ is equivalent to $\|(h_\lambda(B)-p)(h_\lambda(B)-p')^{-1}\| \leqq 1$ (see again Remark 8). But the latter is a consequence of von Neumann's theorem,[5] for $|(h_\lambda(z)-p)(h_\lambda(z)-p')^{-1}| \leqq 1$ for $z \in D$ and $\|B\| \leqq 1$.

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5) See [3].