85. A Construction of Markov Processes by Piecing Out

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In studies of Markov processes we sometimes encounter the situations where we must piece out given Markov processes by an appropriate procedure. Examples are construction of a branching Markov process from a given Markov process which we call the non-branching part and a branching system (cf. [5], [6]), construction of a conservative Markov process from a given process of finite life time (cf. [11]), etc. In this paper we shall discuss such a procedure.

1. Notation and the main theorem. Let S be a locally compact Hausdorff space with countable base and $\bar{S}=S\cup\{\Delta\}$ be the one-point compactification of S (if S is compact Δ is attached as an isolated point).

At first we state the following preliminary

Lemma 1.1. *Let $\{W, \mathcal{B}, P_x, x\in\bar{S}\}$ be a system of probability measures on a σ-field $\mathcal{B}$ of W and let $\mu(w, dy)$ be a probability kernel on $W\times\bar{S}$. Let $\Omega=W\times\bar{S}$, $\mathcal{F}=\mathcal{B}\otimes\mathcal{B}(\bar{S})$, and $\tilde{\Omega}=\prod_{j=1}^{\infty}\Omega_j$, $(\Omega_j=\Omega$, $j=1, 2, \cdots)$ with the product σ-field $\tilde{\mathcal{B}}=\bigotimes_{j=1}^{\infty}\mathcal{F}_j$, $(\mathcal{F}_j=\mathcal{F})$, and put*

$$Q_x(d\omega)=P_x[dw]\mu(w, dy),$$

where we denote $\omega=(w, y)$. Then, there exists a unique probability measure $\tilde{P}_x, (x\in\bar{S})$ on $(\tilde{\Omega}, \tilde{\mathcal{B}})$ satisfying

$$\tilde{P}_x[d\omega^1, d\omega^2, \cdots, d\omega^n]=Q_x(d\omega^1)Q_{x_1}(d\omega^2)\cdots Q_{x_{n-1}}(d\omega^n), \tag{1.1}$$

where $\omega^j=(w_j, x_j)$.

This lemma is a consequence of Ionescu Tulcea's Theorem [7], [9].

For a given right continuous strong Markov process $\{W, x_t, \mathcal{B}_t, \zeta, \theta_t, P_x, x\in S\}$ on $\bar{S}$ with Δ a death point,[1] we define:

Definition 1.1. A kernel $\mu(w, dy)$ defined on $W\times\bar{S}$ will be called an *instantaneous distribution* if it satisfies;

(i) For any fixed $w\in W$, $\mu(w, .)$ is a probability Borel measure on $\bar{S}$, and for any fixed Borel subset A of $\bar{S}$, $\mu(., A)$ is a $\mathcal{N}_\infty$-measurable function on W.[2]

1) i.e. if $x_t(w)=\Delta$ then $x_s(w)=\Delta$ for all $s\geq t$. We set $\zeta(w)=\inf\{t; x_t(w)=\Delta\}$.
2) $\mathcal{N}_t=\mathcal{B}\{x_s; s\leq t\}$, $0\leq t\leq\infty$.

(ii) For $w \in W$ such as $\zeta(w)=0$, $\mu(w, dy)=\delta_{\Delta}(dy)$.
(iii) For any Markov time $T(w)$,

$$P_x[\mu(w, dy)=\mu(\theta_{T(w)}w, dy),\ T(w)<\zeta(w)]=P_x[T(w)<\zeta(w)]. \tag{1.2}$$

In the following we assume that we are given a right continuous strong Markov process $\{W, x_t, \mathcal{B}_t, \zeta, \theta_t, P_x, x \in \bar{S}\}$ on $\bar{S}$ with Δ a death point and an instantaneous distribution $\mu(w, dy)$. And let $\Omega = W \times \bar{S}$, $\tilde{\Omega}$, and $\tilde{P}_x$ be those defined in Lemma 1.1.

Now let $\omega=(w, y) \in \Omega$ we put

$$\dot{x}_t(\omega)=\begin{cases} x_t(w), \text{ if } t<\zeta(w), \\ y, \text{ if } t \geqq \zeta(w), \end{cases} \tag{1.3}$$

and put for $\tilde{\omega}=(\omega^1, \omega^2, \cdots) \in \tilde{\Omega}$,

$$N(\tilde{\omega})=\min\{j;\ \zeta(w_j)=0\},\ (=+\infty, \text{ if such } j \text{ does not exist}). \tag{1.4}$$

We define next $X_t(\tilde{\omega})$ on $\tilde{\Omega}$ by

$$X_t(\tilde{\omega})=\begin{cases} \dot{x}_t(\omega^1), & \text{if } 0 \leqq t \leqq \zeta(w_1), \\ \dot{x}_{t-\zeta(w_1)}(\omega^2), & \text{if } \zeta(w_1)<t \leqq \zeta(w_1)+\zeta(w_2), \\ \cdots\cdots & \\ \dot{x}_{t-(\zeta(w_1)+\cdots+\zeta(w_n))}(\omega^{n+1}), & \text{if } \sum_{j=1}^{n}\zeta(w_j)<t \leqq \sum_{j=1}^{n+1}\zeta(w_j), \\ \cdots\cdots & \\ \Delta, & \text{if } t \geqq \sum_{j=l}^{N(\tilde{\omega})}\zeta(w_j), \end{cases} \tag{1.5}$$

and denote

$$\tau_0(\tilde{\omega})=0,\ \tau(\tilde{\omega})=\tau_1(\tilde{\omega})=\zeta(w_1),\ \cdots,\ \tau_n(\tilde{\omega})=\sum_{j=1}^{n}\zeta(w_j),\ \cdots, \tag{1.6}$$

$$\tilde{\zeta}(\tilde{\omega})=\sum_{j=1}^{N(\tilde{\omega})}\zeta(w_j). \tag{1.7}$$

Lemma 1.2. *Let $\tilde{\Omega}_0=\{\tilde{\omega};\ X_t(\tilde{\omega})$ is right continuous with respect to $t \geqq 0\}$. Then,*

$$\tilde{\Omega}_0=\{\tilde{\omega};\ x_n=x_0(w_{n+1}),\ \textit{for any } n \geqq 1\},\ \textit{and}$$

$$\tilde{P}_x[\tilde{\Omega}_0]=1, \qquad x \in S. \tag{1.8}$$

Therefore, we can restrict every quantities defined on $\tilde{\Omega}$ to $\tilde{\Omega}_0$.

The shift operator θ_t of $\tilde{\omega} \in \tilde{\Omega}_0$ is defined as

$$\theta_t\tilde{\omega}=((\theta_{t-\tau_k(\tilde{\omega})}w_{k+1}, x_{k+1}), \omega^{k+2}, \cdots), \text{ if } \tau_k(\tilde{\omega}) \leqq t<\tau_{k+1}(\tilde{\omega}), \tag{1.9}$$

where $\tilde{\omega}=(\omega^1, \omega^2, \cdots)$ and $\omega^j=(w_j, x_j)$, $j=1, 2, \cdots$.

Let φ_k be the projection from $\tilde{\Omega}_0$ to $\prod_{j=1}^{k}\Omega_j$, $(\Omega_j=\Omega)$ and define

$$\begin{gathered} \mathcal{B}_{\tau_k}=\varphi_k^{-1}(\bigotimes_{j=1}^{k}\mathcal{F}_j) \cap \tilde{\Omega}_0, \text{ where } \mathcal{F}_j=\mathcal{N}_\infty \otimes \mathcal{B}(S), \\ \tilde{\mathcal{B}}=\bigvee_{k=1}^{\infty}\mathcal{B}_{\tau_k}=\bigotimes_{j=1}^{\infty}\mathcal{F}_j \cap \tilde{\Omega}_0, \text{ and} \\ \tilde{\mathcal{N}}_t=\mathcal{B}\{X_s;\ \forall s \leqq t\} \cap \tilde{\Omega}_0. \end{gathered} \tag{1.10}$$

Definition 1.2. $\tilde{\omega}, \tilde{\omega}' \in \tilde{\Omega}_0$ is said to be *R_t-equivalent* and we denote

$$\tilde{\omega} \sim \tilde{\omega}' \quad (R_t),$$

if;

(i) $X_s(\tilde{\omega})=X_s(\tilde{\omega}')$, for any $s\leqq t$, and
(ii) if $\tau_k(\tilde{\omega})\leqq t<\tau_{k+1}(\tilde{\omega})$, then $\tau_k(\tilde{\omega}')\leqq t<\tau_{k+1}(\tilde{\omega}')$ and $\tau_j(\tilde{\omega})=\tau_j(\tilde{\omega}')$ for any $j\leqq k$.

Now we define new σ-field $\tilde{\mathscr{B}}_t$ by

(1.11) $\tilde{\mathscr{B}}_t=\{A;$ i) $A\in\tilde{\mathscr{B}}$, and ii) if $\tilde{\omega}\in A$ and $\tilde{\omega}\sim\tilde{\omega}'(R_t)$, then $\tilde{\omega}', \in A\}$.

It is clear that $\tilde{\mathscr{B}}_t$ is a σ-field of $\tilde{\Omega}_0$ and $\tilde{\mathfrak{N}}_t\subset\tilde{\mathscr{B}}_t$.

Remark. (i) $\tau_k(\tilde{\omega})$ is a $\tilde{\mathscr{B}}_t$-Markov time but it is not necessarily $\tilde{\mathfrak{N}}_t$-Markov time.

(ii) If we put $\tilde{\mathscr{B}}_\infty=\bigvee_{t>0}\tilde{\mathscr{B}}_t$, then $\tilde{\mathscr{B}}_\infty=\tilde{\mathscr{B}}$.

(iii) Put $\tilde{\mathscr{B}}_{\tau_k}=\{A; A\in\tilde{\mathscr{B}}$, and $A\cap\{\tau_k<t\}\in\tilde{\mathscr{B}}_t$ for any $t\geqq 0\}$, then $\tilde{\mathscr{B}}_{\tau_k}=\mathscr{B}_{\tau_k+}$.

Now our main Theorem is stated as follows.

Theorem 1.1. *Let $\{W, x_t, \mathscr{B}_t, \zeta, \theta_t, P_x, x\in S\}$ be a right continuous strong Markov process on $\bar{S}$ with Δ as a death point and $\mu(w, dy)$ be an instantaneous distribution. Then, the above defined system $X=\{\tilde{\Omega}_0, X_t, \tilde{\mathscr{B}}_t, \tilde{\zeta}, \theta_t, \tilde{P}_x, x\in S\}$ is a right continuous strong Markov process on $\bar{S}$, where $\tilde{P}_\Delta[X_t=\Delta, {}^{\forall}t\geqq 0]=1.$*[3]

For the proof, we need several lemmas.

2. Lemmas. We first note that for any $\tilde{\mathscr{B}}_t$-Markov time $T(\tilde{\omega})$ Galmariono's test[4] remains valid, i.e.,

Lemma 2.1. *For any $t\geqq 0$, random time $T(\tilde{\omega})\geqq 0$ satisfies*

$$\{\tilde{\omega};\, T(\tilde{\omega})<t\}\in\tilde{\mathscr{B}}_t, \quad (\{\tilde{\omega};\, T(\tilde{\omega})\leqq t\}\in\tilde{\mathscr{B}}_t),$$

if and only if (i) *$T(\tilde{\omega})$ is $\tilde{\mathscr{B}}$-measurable and* (ii) *if $T(\tilde{\omega})<t$ (resp. $T(\tilde{\omega})\leqq t$) and $\tilde{\omega}\sim\tilde{\omega}'(R_t)$ then $T(\tilde{\omega})=T(\tilde{\omega}')$.*

Lemma 2.2. $\tilde{\mathscr{B}}_\infty=\tilde{\mathscr{B}}_t\vee\theta_t^{-1}(\tilde{\mathscr{B}}_\infty)$.

Making a slight modification, Courrége-Priouret's results [1] are valid in our case, i.e.,

Lemma 2.3. *Let $T(\tilde{\omega})$ be a $\tilde{\mathscr{B}}_t$-Markov time and take any integer k. Then there exists $T_k(\tilde{\omega}, \tilde{\omega}')$ on $\tilde{\Omega}_0\times\tilde{\Omega}_0$ satisfying*

1) *$T_k(\tilde{\omega}, \tilde{\omega}')$ is $\tilde{\mathscr{B}}_{\tau_k}\otimes\tilde{\mathscr{B}}_\infty$-measurable,*
2) *for fixed $\tilde{\omega}$, $T_k(\tilde{\omega}, .)$ is $\tilde{\mathscr{B}}_t$-Markov time, and*
3) $T(\tilde{\omega})\vee\tau_k(\tilde{\omega})=\tau_k(\tilde{\omega})+T_k(\tilde{\omega}, \theta_{\tau_k(\tilde{\omega})}\tilde{\omega})$.

If we notice the way how the measure $\tilde{P}_x$ and the random variable X_t were constructed and the properties of the instantaneous distribution, we are able to verify the following

Lemma 2.4. (i) *For any $B\in\tilde{\mathscr{B}}$ and $A\in\tilde{\mathscr{B}}_{\tau_k}$,*

(2.1) $$\tilde{P}_x[\theta_{\tau_k}\tilde{\omega}\in B, A]=\tilde{E}_x[\tilde{P}_{X_{\tau_k}}[B]; A].$$

3) If x_t is merely Markov, then X_t is also Markov. Of course, X_t is temporally homogeneous.

4) Cf. [4].

(ii) *Let* $g(\tilde{\omega}, t)$ *be a bounded measurable function on* $\tilde{\Omega}_0\times[0, \infty]$ *and* $\sigma(\tilde{\omega})$ *be* $\tilde{\mathscr{B}}_{\tau_k}$*-measurable, then for any* $A\in\tilde{\mathscr{B}}_{\tau_k}$,

$$(2.2)\qquad \tilde{E}_x[g(\theta_{\tau_k}\tilde{\omega}, \sigma(\tilde{\omega})); A]=\tilde{E}_x[\tilde{E}_{X_{\tau_k}}[g(\cdot, s)]\,|_{s=\sigma}; A].$$

(iii) *Let* $g(\tilde{\omega}, \tilde{\omega}')$ *be a bounded* $\tilde{\mathscr{B}}_{\tau_k}\otimes\tilde{\mathscr{B}}$*-measurable function, then for any* $A\in\tilde{\mathscr{B}}_{\tau_k}$,

$$(2.3)\qquad \tilde{E}_x[g(\tilde{\omega}, \theta_{\tau_k}\tilde{\omega}); A]=\tilde{E}_x[\tilde{E}_{X_{\tau_k}}[g(u, .)]\,|_{u=\tilde{\cdot}}; A].$$

Lemma 2.5. *Let* $T(\tilde{\omega})$ *be a* $\tilde{\mathscr{B}}_t$*-Markov time, then there exists a* $\mathscr{B}_t$*-Markov time* $T(w)$ *defined on* W, *such as*

$$T(\tilde{\omega})=T(w), \text{ on } \{T<\tau\},$$

where $\tilde{\omega}=((w, y), \omega^2, \omega^3, \cdots)$.

Lemma 2.6. *Let* $f(x)$ *and* $g(x, t)$ *be bounded measurable functions on* S *and* $S\times[0, \infty]$, *then for any* $\tilde{\mathscr{B}}_t$*-Markov time* $T(\tilde{\omega})$,

$$(2.4)\quad \tilde{E}_x[f(X_T)g(X_\tau, \tau-T); T<\tau]=\tilde{E}_x[f(X_T)\tilde{E}_{X_T}[g(X_\tau, \tau)]; T<\tau].$$

Lemma 2.7. *Let* $g(x, t)$ *be a bounded measurable function on* $S\times[0, \infty]$ *and* $T(\tilde{\omega})$ *be any* $\tilde{\mathscr{B}}_t$*-Markov time, then for any* $A\in\tilde{\mathscr{B}}_T$,

$$(2.5)\qquad \tilde{E}_x[g(X_{\tau(\theta_T\tilde{\omega})}(\theta_T\tilde{\omega}), \tau(\theta_T\tilde{\omega})); A]=\tilde{E}_x[\tilde{E}_{X_T}[g(X_\tau, \tau)]; A].$$

3. Proof of Theorem 1.1. Let $f(x)$ be a bounded measurable function on S for which we put $f(\Delta)=0$, $T(\tilde{\omega})$ be a $\tilde{\mathscr{B}}_t$-Markov time and $A\in\tilde{\mathscr{B}}_T$. In order to prove Theorem 1.1, it is sufficient for us to show

$$(3.1)\qquad \tilde{E}_x[f(X_{T+t}); A]=\tilde{E}_x[\tilde{E}_{X_T}[f(X_t)]; A].$$

This is verified by means of the above mentioned Lemmas. We shall sketch the proof.

Put

$$\mathrm{I}=\tilde{E}_x[f(X_{T+t}); A\cap\{\tilde{\omega}; T(\tilde{\omega})<\tau_k(\tilde{\omega})\leqq T(\tilde{\omega})+t, \text{ for some } k\}],$$

and

$$\mathrm{II}=\tilde{E}_x[f(X_{T+t}); A\cap\{\tilde{\omega}; \tau_k(\tilde{\omega})\leqq T(\tilde{\omega}), T(\tilde{\omega})+t<\tau_{k+1}(\tilde{\omega}), \text{ for some } k\}].$$

If we notice

$$\begin{aligned}(3.2)\quad &\tilde{E}_x[f(X_{T+t}); \tau_k\leqq T, T+t<\tau_{k+1}, A]\\ &=\tilde{E}_x[\chi(\tau_k\leqq T)\tilde{E}_{X_{\tau_k}}[f(X_{T_k(u,\cdot)+t}); 0\leqq T_k(u, \cdot)<\tau,\\ &\qquad\qquad 0\leqq T_k(u, \cdot)+t<\tau]\,|_{u=\tilde{\omega}}; A]\\ &=\tilde{E}_x[\chi(\tau_k\leqq T)\tilde{E}_{X_{\tau_k}}[\tilde{E}_{X_{T_k(u,\cdot)}}[f(X_t); 0\leqq t<\tau];\\ &\qquad\qquad 0\leqq T_k(u, \cdot)<\tau]\,|_{u=\tilde{\omega}}; A]\\ &=\tilde{E}_x[\chi(\tau_k\leqq T<\tau_{k+1})\tilde{E}_{X_T}[f(X_t); 0\leqq t<\tau]; A],\end{aligned}$$ [5]

we have

$$\begin{aligned}(3.3)\qquad \mathrm{II}&=\sum_{k=0}^{\infty}\tilde{E}_x[f(X_{T+t}); \tau_k\leqq T, T+t<\tau_{k+1}; A]\\ &=\tilde{E}_x[\tilde{E}_{X_T}[f(X_t); 0\leqq t<\tau]; A].\end{aligned}$$

Therefore we have

$$\tilde{E}_x[\tilde{E}_{X_T}[f(X_t)]; A]-\mathrm{II}=\tilde{E}_x[\tilde{E}_{X_T}[f(X_t); \tau\leqq t]; A].$$

5) $\chi(A)$ is the indicator of a set A.

Thus it is sufficient for us to show

(3.4) $$\mathrm{I}=\tilde{E}_x[\tilde{E}_{X_T}[f(X_t);\ \tau\leqq t];\ A],$$

but this is verified as follows:

$$\begin{aligned}
&\tilde{E}_x[\tilde{E}_{X_T}[f(X_t);\ \tau\leqq t];\ A]\\
&=\tilde{E}_x[\tilde{E}_{X_T}[\tilde{E}_{X_\tau}[f(X_{t-s});\ t-s\geqq 0]\,|_{s=\tau};\ \tau\leqq t];\ A]\\
&=\tilde{E}_x[\tilde{E}_{X_{\tau(\theta_T\tilde{\omega})}(\theta_T\tilde{\omega})}[f(X_{t-s});\ t-s\geqq 0]\,|_{s=\tau(\theta_T\tilde{\omega})};\ \tau(\theta_T\tilde{\omega})\leqq t,\ A]\\
&=\sum_{k=0}^{\infty}\tilde{E}_x[\chi(\tau_k\leqq T<\tau_{k+1})f(X_{t+T-\tau_{k+1}}(\theta_{\tau_{k+1}}\tilde{\omega}));\ \tau_{k+1}-T\leqq t,\ A]\\
&=\mathrm{I}.
\end{aligned}$$

4. Some properties of the process X_t. Let $\tilde{\mathcal{B}}_t(\tilde{P}_x)$ be the completion of $\tilde{\mathcal{B}}_t$ with respect to $\tilde{P}_x$, and put

$$\tilde{\mathcal{F}}_t=\bigcap_{x\in S}\tilde{\mathcal{B}}_t(\tilde{P}_x),$$

then, we have

Theorem 1.1′. *Under the same notations of Theorem* 1.1, $\{\tilde{\Omega}_0, X_t, \tilde{\mathcal{F}}_t, \tilde{\zeta}, \theta_t, \tilde{P}_x, x\in\bar{S}\}$ *is a right continuous strong Markov process.* (Cf. [12], [2]).

Proposition 4.1. *If $x_t(w)$ has the left limit at $t\in(0,\zeta(w)]$, P_x-a.e., then $X_t(\tilde{\omega})$ has the left limit at $t\in(0,\tilde{\zeta}(\tilde{\omega}))$, $\tilde{P}_x$-a.e.*

Proposition 4.2. *If $x_t(w)$ is quasi-left continuous and $\zeta(w)$ is non-accessible (i.e. totally inaccessible in the strong sense in the sense of Meyer [10]), then $X_t(\tilde{\omega})$ is quasi-left continuous before $\tilde{\zeta}(\tilde{\omega})$, i.e., for any sequence of Markov times $T_n\uparrow T$,*

$$\tilde{P}_x[\lim_{n\to\infty}X_{T_n}=X_T;\ T<\tilde{\zeta}]=\tilde{P}_x[T<\tilde{\zeta}].$$

Corollary. *If $x_t(w)$ is a Hunt process and $\zeta(w)$ is non-accessible, and if $\tilde{P}_x[\tilde{\zeta}=\infty]=1$, then $X_t(\tilde{\omega})$ is a Hunt process.*

Proposition 4.3. *Let the instantaneous distribution $\mu(w,dy)$ be a probability measure on S for such w that $\zeta(w)>0$, and $x_t(w)$ satisfy either*

(i) $$\sup_{x\in S}P_x[\zeta<\infty]=a<1, \text{ or}$$

(ii) *there exist $\varepsilon>0$ and $\delta>0$ such as*

$$\inf_{x\in S}P_x[\zeta>\varepsilon]>\delta.$$

Then, X_t is conservative i.e.,

$$\tilde{P}_x[\tilde{\zeta}=\infty]=1,\ (x\in S).$$

5. Applications. i) Let $X_t(w)$ satisfy $P_x[\exists x_{\zeta-}\in S]=1$, $x\in S$, and $\mu'(x,dy)$ be a probability kernel on $S\times S$. Put

$$\mu(w,dy)=\mu'(x_{\zeta-}(w),dy),\ \text{and}\ \mu(w_\Delta,dy)=\delta_\Delta(dy),\text{[6]}$$

then $\mu(w,dy)$ is an instantaneous distribution. In particular, if we take

$$\mu'(x,dy)=\delta_x(dy),$$

6) $x_t(w_\Delta)=\Delta$ for all $t\geqq 0$.

Theorem 1.1 reduces to the case treated in [11].

ii) Let S have a boundary ∂S in some sense, and given a kernel $\mu'(x, dy)$ on $\{S \cup \partial S\} \times S$ and a Markov process $x_t(w)$ on $S \cup \partial S$ with $P_x[\exists x_{\zeta-} \in S \cup \partial S] = 1$. Put

$$\mu(w, dy) = \mu'(x_{\zeta-}, dy),$$

and apply Theorem 1.1, then we have a process so-called with instantaneous return from the boundary ∂S (cf. [8], [3]).

iii) Theorem 1.1 is applicable to construction of a branching Markov process. But since it needs some preparatory consideration, we will treat it in the forthcoming paper.

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