134. *Operators of Discrete Analytic Functions and Their Application*

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Introduction. In the previous paper [4] we have studied basic properties of operators of discrete analytic functions. In this paper we shall study the uniform convergence of sequences of operators and show operational solutions of a discrete Volterra integral equation and a linear discrete derivative equation by making use of operators of discrete analytic functions.

1. Uniform convergence of sequences of operators. The set A of all discrete analytic functions is a linear space of infinite dimension. By the norm $\|f\|$ of $f \in A$ we understand the number $\|f\| = \sup |f(x, y)|$, where (x, y) is a finite lattice point in the first quadrant.

By the norm, the *uniform convergence* in A is defined as follows: A sequence f_n of A converges uniformly to an element f of A if and only if the sequence $\|f_n - f\|$ tends to 0 as $n \to \infty$. The convergence is denoted by

$$\lim_A f_n = f.$$

The normed space A is *complete*, i.e. any Cauchy sequence is convergent.

Thus A is a *Banach space.*

Theorem 1.1. *If f_n and $g_n \in A$, and $\lim_A f_n = f$, $\lim_A g_n = g$, then $\lim_A (f_n * g_n) = f * g$.*

This means that $*$ is *continuous* in the *norm topology.*

A sequence of operators a_n is said to be *convergent in* $\boldsymbol{Op}$, if divided by a suitably chosen operator q, it becomes a sequence of functions $\in A$ uniformly convergent to $f \in A$. Then we have

$$\lim_{Op} a_n = q \lim_A \left(\frac{a_n}{q}\right). \tag{1.1}$$

Theorem 1.2. *If $\lim_{Op} a_n = a$, $\lim_{Op} b_n = b$, then*

$$\lim_{Op} (a_n \pm b_n) = a \pm b, \ \lim_{Op} (a_n b_n) = ab. \tag{1.2}$$

Theorem 1.3. *Let a be a complex number. The power series*

$$\sum_{n=0}^{\infty}{}' \frac{a^n z^{(n)}}{n!} \tag{1.3}$$

converges uniformly in any bounded domain in the first quadrant.

The limit function is called *pseudo exponential function.* The function (1.3) is denoted by e^{az*}.

Theorem 1.4. *An operator corresponding to e^{az*} is*

$$e^{az*}=\frac{1}{s-a}. \tag{1.4}$$

A *rational operator*

$$F(s)=\frac{b_0s^m+b_1s^{m-1}+\cdots+b_m}{a_0s^n+a_1s^{n-1}+\cdots+a_n} \quad (m<n,\ a_0\neq 0) \tag{1.5}$$

can be decomposed into simple fractions of the type $\frac{b}{(s-a)^k}$.

Making use of (1.4), we have

$$F(s)=\sum_{i,j} b_{i,j}(e^{a_jz*})^{*k_j} \tag{1.6}$$

Theorem 1.5. *If the radius of convergence of the series $\sum_{r=0}^{\infty} a_r\lambda^r$ is positive, then $\sum_{r=0}^{\infty} a_rf^{*r}(f\in \boldsymbol{A})$ is uniformly convergent.*

For example, the *operator series*

$$\frac{1}{1-f}=1+f+f^{*2}+\cdots+f^{*n}+\cdots$$

holds true.

2. Applications. R. J. Duffin and C. S. Duris have solved in [2] the discrete Volterra integral equation

$$u(z)=f(z)+\lambda\int_0^z k(z-t):u(t)\delta t, \tag{2.1}$$

where $f(z)\in \boldsymbol{A}$, $k(z)\in \boldsymbol{A}$, and λ is a constant.

We can solve it operationally in a simpler way.

The integral equation (2.1) is written in the form of

$$u(z)*\{1-\lambda k(z)\}=f(z), \tag{2.2}$$

where 1 is a function corresponding to an operator [1]. Therefore $u(z)\in \boldsymbol{A}$ is uniquely determined for the given functions $f(z)$ and $k(z)$, by Theorem 2.1 in [4], if

$$\lambda\bar{k}_{1,0}\neq 2, \text{ and } \lambda\bar{k}_{0,1}\neq 2/p.$$

From (2.2) we have the operational equation

$$u(1-\lambda k)=f.$$

Hence, we have

$$u=f+\lambda k*f+\lambda^2k^{*2}*f+\cdots+\lambda^nk^{*n}*f+\cdots.$$

R. J. Duffin and C. S. Duris have solved in [2] a linear discrete derivative equation with constant coefficients.

We deal with a linear discrete derivative equation with pseudo polynomial coefficients of convolutional product, containing the equation with constant coefficients as a special case.

If

$$F(z)=\int_a^z f(t)\delta t+F(a),$$

then $F(z) \in \boldsymbol{A}$ follows from $f(z) \in \boldsymbol{A}$, and the converse holds. In this case, $f(z)$ is called a *discrete derivative* of $F(z)$ and is denoted by

$$f(z)=\frac{\delta F}{\delta z}.$$

Therefore we have

$$\int_0^z \frac{\delta F}{\delta z}\,\delta t = F(z)-F(0).$$

This is expressed in $\boldsymbol{Op}$ by

$$l\frac{\delta F}{\delta z}=F-[F(0)]l.$$

Therefore we have

$$\frac{\delta F}{\delta z}=sF-[F(0)],$$

$$\frac{\delta^2 F}{\delta z^2}=s^2F-s[F(0)]-\left[\frac{\delta F(0)}{\delta z}\right],$$

....

$$\frac{\delta^n F}{\delta z^n}=s^nF-s^{n-1}[F(0)]-s^{n-2}\left[\frac{\delta F(0)}{\delta z}\right]-\cdots-s\left[\frac{\delta^{n-2}F(0)}{\delta z^{n-2}}\right]-\left[\frac{\delta^{n-1}F(0)}{\delta z^{n-1}}\right].$$

We now consider the problem of finding $F(z)$ such that

$$k_0(z^*)*\frac{\delta^n F(z)}{\delta z^n}+k_1(z^*)*\frac{\delta^{n-1}F(z)}{\delta z^{n-1}}+\cdots+k_n(z^*)*F(z)=f(z),$$

where $k_i(z^*)(i=0,1,\cdots,n)$ are pseudo polynomials of the type $\sum a_k z^{(k)}$.

This is expressed in $\boldsymbol{Op}$ by

$$k_0\left(s^nF-s^{n-1}[F(0)]-\cdots-\left[\frac{\delta^{n-1}F(0)}{\delta z^{n-1}}\right]\right)$$

$$+k_1\left(s^{n-1}F-s^{n-2}[F(0)]-\cdots-\left[\frac{\delta^{n-2}F(0)}{\delta z^{n-2}}\right]\right)+\cdots+k_nF=f.$$

Hence we have

$$F(k_0s^n+k_1s^{n-1}+\cdots+k_n)-(k_0s^{n-1}+k_1s^{n-2}+\cdots+k_{n-1})[F(0)]$$

$$-\cdots-k_0\left[\frac{\delta^{n-1}F(0)}{\delta z^{n-1}}\right]=f.$$

Put

$$k_i(z^*)=\sum_{j=1}^{m_i} a_{i,j}l^j,$$

and making use of (1.5), (1.6), we can solve the above equation operationally.

The theory of discrete analytic functions of two integral variables will be extended to the theory of functions of n integral variables.

The detailed proofs of the results obtained in this paper will be published in [3].

References

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