197. Local Estimations of Conjugate Functions. I

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1966)

1. Let $f(x)$ be a measurable and 2π-periodic function which is defined on the real line. The conjugate function of $f(x)$ is defined by the following formula:

$$(1.1)\qquad \tilde{f}(x)=\text{P.V.}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{2}\cot\frac{1}{2}(x-t)dt$$
$$=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon} f(t)\frac{1}{2}\cot\frac{1}{2}(x-t)dt.$$

As everybody know, there are results due to M. Riesz, A. N. Kolmogorov, A. Zygmund and others [7, Chap. VII, XII]. As well as Hilbert tranforms it is understood to be singular integral operator of convolution type. The Hilbert transform of $f(x)$ is defined by the following formula:

$$(1.2)\qquad \tilde{f}(x)=\text{P.V.}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{dt}{x-t}$$
$$=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon} f(t)\frac{dt}{x-t}$$

where periodicity of f does not assumed. It is extended to many variable case by A. P. Calderon and A. Zygmund [5] and others. Besides, these operators have a characteristic properties. That is, it is invariant under dilatation and translation. From this stand point of view, there are interesting studies due to M. Cotlar [1] and L. Hörmander [3].

On the other hand, there is another interesting problem. That is as follows: define the good intergal which has the following property, if f is integrable in certain sense, then its conjugate function exists and is integrable in the same sense. To this problem we should need local integrability of conjugate function.

2. Local estimations of conjugate functions on sub-interval. The purpose of this section is to prove the following result.

Theorem 1. *Let f belong to the class $Lp\,(p>1)$. Let us denote by $I=(a, b)$ a sub-interval of $(-\pi, \pi)$, and by $I'=(a', b')$ the enlarged interval of I into two times symmetrically. Then we have*

$$(2.1)\qquad \sup_{|x|\leq\pi}\left(\int_a^b|\tilde{f}(x+t)|^p dt\right)^{\frac{1}{p}}\leq A_p\log\delta^{-1}\sup_{|x|\leq\pi}\left(\int_a^b|f(x+t)|^p dt\right)^{\frac{1}{p}}$$

where $A_p=O(1/p-1)$ as $p\to 1$ and $\delta=b-a$.

Proof. By the periodicity we can assume that $I'=(a', b')$ is contained in $(-\pi, \pi)$ without loss of generality. Let us introduce conjugate functions of the interval $I'=(a', b')$, That is

$$\tilde{f}_{I'}(x)=-\frac{1}{b'-a'}\int_{a'}^{b'} f(t)\cot\frac{\pi}{(b'-a')}(t-x)dt.$$

Then we have by the M. Riesz theorem

$$\left(\int_{a'}^{b'}|f_{I'}(x)|^p\,dx\right)^{\frac{1}{p}}\leqq A_p\left(\int_{a'}^{b'}|f(x)|^p\,dx\right)^{\frac{1}{p}}\qquad (p>1)$$

where $A_p=O(1/p-1)$ as $p\to 1$. Then we can write

$$\begin{aligned}&\tilde{f}(x)-\tilde{f}_{I'}(x)\\&=-\int_{a'}^{b'} f(t)\left\{\frac{1}{2\pi}\cot\frac{1}{2}(t-x)-\frac{1}{b'-a'}\cot\frac{\pi}{(b'-a')}(t-x)\right\}dt\\&\quad-\frac{1}{\pi}\left(\int_{-\pi}^{a'}+\int_{b'}^{\pi}\right)f(t)\frac{1}{2}\cot\frac{1}{2}(t-x)\,dt\\&=J_1+J_2+J_3,\ \text{say}.\end{aligned}$$

As for J_1, the property of kernel

$$K(t, x)=\frac{1}{2\pi}\cot\frac{1}{2}(t-x)-\frac{1}{b'-a'}\cot\frac{\pi}{(b'-a')}(t-x)=O(t-x)/(b'-a')^2$$

reads the following by applying the Hölder inequality,

$$\begin{aligned}|J_1|&\leqq\int_a^b\left|\int_{a'}^{b'} f(t)K(t, x)\,dt\right|^p dx\leqq\int_a^b\int_{a'}^{b'}|f|^p dt\left(\int_{a'}^{b'}|K|^q dt\right)^{\frac{p}{q}}dx\\&=O\frac{(b-a)}{(b'-a')}2p\left(\int_0^{(b'-a')}u^q du\right)^{\frac{p}{q}}\int_{a'}^{b'}|f|^p dt=O\left(\int_a^b|f|^p dt\right)\end{aligned}$$

Next as for J_2, we have by decomposing into small pieces,

$$\begin{aligned}|J_2|&\leqq\int_a^b\left|-\frac{1}{\pi}\int_{-\pi}^{a'} f(t)\frac{1}{2}\cot\frac{1}{2}(t-x)dt\right|^p dx\\&\leqq\int_a^b\int_{-\pi}^{a'}\frac{|f(t)|^p}{|t-x|}dt\left(\int_{-\pi}^{a'}\frac{dt}{|t-x|}\right)^{\frac{p}{q}}dx\\&=O(\log\delta^{-1})^{\frac{p}{q}}\int_a^b\left(\sum_{j=0}^{N}\int_{a'-(j+1)\delta}^{a'-j\delta}|f|^p\frac{dt}{|a'+j\delta-a|}\right)dx\end{aligned}$$

where $N=\left[\frac{a'-(-\pi)}{\delta}\right]+1=O(\delta^{-1})$, if we denote by I_1 the interval such as $I_1\subset(-\pi, a')$ and $|I_1|=\delta$, then we have

$$\begin{aligned}|J_2|&\leqq O\frac{(\log\delta^{-1})^{\frac{p}{q}}}{\delta}\left(\int_a^b dx\right)\sup_{I_1}\int_{I_1}|f(t)|^p dt\left(\sum_{j=1}^{N}\frac{1}{j+1}\right)\\&=O(\log\delta^{-1})^p\sup_{I_1}\int_{I_1}|f(t)|^p dt\end{aligned}$$

Similarly we have as for J_3,

$$|J_3|\leqq O(\log\delta^{-1})^p\sup_{I_2}\int_{I_2}|f(t)|^p dt$$

where I_2 is the interval such as $I_2\subset(b', \pi)$ and $|I_2|=\delta$. From these evaluations we obtain

$$\int_a^b |\tilde{f}(x)|^p dx \leqq A_p^p \int_{a'}^{b'} |f(t)|^p dt + 2B(\log \delta^{-1})^p \sup_{I_3} \int_{I_3} |f(t)|^p dt$$

where I_3 is the interval such as $I_3 \subset (-\pi, a') \cup (b', \pi)$ and $|I_3| = \delta$. Thus we have proved the theorem.

We have two corollaries by running the same line

Theorem 2. *Let f belong to the $L \log^+ L$ class, that is*

$$\int_{-\pi}^{\pi} |f(x)| \log^+ |f(x)| dx$$

exist and finite. Then we have

$$\sup_{|x| \leqq \pi} \int_a^b |\tilde{f}(x+t)| dt \leqq A(\log \delta^{-1}) \sup_{|x| \leqq \pi} \int_a^b |f(x+t)| \log^+ |f(x+t)| dt + B \tag{2.2}$$

where A and B are absolute constants and $B = O(\delta)$.

Theorem 3. *Let f belong to the calss L, then we have*

$$\sup_{|x| \leqq \pi} |\{t \mid |\tilde{f}(x+t)| > y,\ t+x \in (a, b)\}| \leqq \frac{A \log \delta^{-1}}{y} \sup_{|x| \leqq \pi} \int_a^b |f(x+t)| dt \tag{2.3}$$

where A is an absolute constant.

3. Local estimations of conjugate functions on measurable sub-set. The purpose of this section is to prove the following theorem.

Theorem 4. *Let f belong to the calss $L \log^+ L$. Then for any measurable set E in $(-\pi, \pi)$ and $|E| \leqq e$ we have*

$$\sup_{|E| \leqq e} \int_E |\tilde{f}(t)| dt \leqq A(\log \delta^{-1}) \sup_{|E| \leqq e} \int_E |f(t)| \log^+ |f(t)| dt + B \tag{3.1}$$

where A and B are absolute constants and $B = 0(\delta)$.

We shall quote the Stein-Weiss lemma (4).

Lemma (E. M. Stein-G. Weiss). *Let $E \subset (-\pi, \pi)$ be a measurable set, $\tilde{\chi}_E$ the conjugate function of its characteristic function χ_E and $\lambda(y)$, $y > 0$, the distribution function of $|\tilde{\chi}_E|$, then $\lambda(y)$ satisfies the equation*

$$\exp\left\{\frac{1}{2} i\lambda(y)\right\} = \frac{\sinh \frac{1}{2} y + i \sin \frac{1}{2} |E|}{\sinh \frac{1}{2} y - i \sin \frac{1}{2} |E|}$$

consequently there exists an absolute constant $A > 0$ such that

$$\lambda(y) < \frac{A}{\sinh \frac{1}{2} y} |E|.$$

They carried a new proof of the M. Riesz theorem applying this lemma. The author learned it from Prof. A. Zygmund [6] of his lecture in Tokyo.

Proof. The proof can by done running on the same lines. Let us put for every measurable set E such as $|E| \leqq e$, $E_1 = \{t \mid \tilde{f}(t) > 0\} \cap E$, $E_2 = \{t \mid \tilde{f}(t) < 0\} \cap E$. Then we have

$$\int_E |\tilde{f}|dt=\int_{E_1}\tilde{f}dt-\int_{E_2}\tilde{f}dt=\int_0^{2\pi}\tilde{f}\chi_{E_1}dt-\int_0^{2\pi}\tilde{f}\chi_{E_2}dt$$
$$=-\int_0^{2\pi}f\tilde{\chi}_{E_1}dt+\int_0^{2\pi}f\tilde{\chi}_{E_2}dt$$

We have also

$$\int_E |\tilde{f}|dt\leqq\int_0^{2\pi}|f||\tilde{\chi}_{E_1}|dt+\int_0^{2\pi}|f||\tilde{\chi}_{E_2}|dt$$
$$\leqq\int_0^{2\pi}f^*\tilde{\chi}^*_{E_1}dt+\int_0^{2\pi}f^*\tilde{\chi}^*_{E_2}dt$$

where f^* is the decreasing rearrangement of f and the same as for $\tilde{\chi}^*_{E_1}$ and $\tilde{\chi}^*_{E_2}$. By the Stein-Weiss lemma we have

$$|\tilde{\chi}^*_{E_i}|\leqq 2\sinh^{-1}Ae_i/t,\ |E_i|=e_i\quad (i=1,2).$$

Therefore we have

$$\int_E |\tilde{f}|dt\leqq 4\int_0^{2\pi}f^*(t)\sinh^{-1}Ae/t\,dt$$
$$=4\left(\int_0^e+\int_e^{2\pi}\right)(\quad)dt=4(J_1+J_2),\ \text{say}.$$

Let us notice the following properties

$$\sinh^{-1}t=O(\log(2t+1)),\quad 0<t<\infty$$

and

$$(\sinh^{-1}t)'=1/\sqrt{1+t^2}$$

As for J_1, if we put $F(t)=\int_0^t f^*(u)du$, integrating by parts and applying the Hardy-Littlewood maximal theorem we obtain

$$J_1=\int_0^e f^*\sinh^{-1}Ae/tdt=[F(t)\sinh^{-1}Ae/t]_0^e+\int_0^e\frac{F(t)}{t}\cdot\frac{Ae}{\sqrt{(Ae)^2+t^2}}dt$$
$$=O(F(e))+\int_0^e\left(\frac{1}{t}\int_0^t f^*(u)du\right)dt$$
$$J_1\leqq O\left(\int_0^e f^*(t)dt\right)+\int_0^e f^*(t)\log^+f^*(t)dt+B,$$

where $B=O(e)$. As for J_2, we have by decomposing into small pieces,

$$J_2=\int_e^{2\pi}f^*(t)\sinh^{-1}Ae/t\,dt=O\left(\int_e^{2\pi}f^*(t)\log\left(\frac{2Ae}{t}+1\right)dt\right)$$
$$=O\left(\sum_{j=1}^N\int_{je}^{(j+1)e}f^*(t)\log\left(\frac{2Ae}{t}+1\right)dt\right)$$
$$\leqq\int_0^e f^*(t)dt\cdot O\left(\sum_{j=1}^N\log\left(\frac{2A}{j}+1\right)\right)$$

where $N=O(2\pi/e)$ and we obtain

$$J_2\leqq 0(A\log N)\int_0^e f^*(t)dt=O(A\log e^{-1})\int_0^e f^*(t)dt.$$

Combining these evaluations we obtain the theorem.

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