193. On Axiom Systems of Propositional Calculi. XXIII

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In our papers ([1], [5]), by using J. Lukasiewicz method, we proved that the Russell system:

1 $CpCqp$,

2 $CCpqCCqrCpr$,

3 $CCpCqrCqCpr$,

4 $CNNpp$,

5 $CCpNpNp$,

6 $CCpNqCqNp$

is equivalent to the classical propositional calculus.

In my paper [2], the propositional calculus satisfying the conditions 1–3, 5 and 6 mentioned above is called a *NB-system*. For any implicational calculus not containing the negation functor N, we introduce the symbol '0' as a propositional constant, and define Np as $Cp0$ (for details, see [4], pp. 50–51).

As well known, an axiom system of the positive implicational calculus is given by J. Lukasiewicz as follows:

7 $CpCqp$,

8 $CCpCqrCCpqCpr$.

In our paper [1], we deduced some theses from 7 and 8. For example, we proved the following theses:

9 $CCpCqrCqCpr$,

10 $CCpqCCqrCpr$,

11 $CCpCpqCpq$.

We define

12 $Np = Cp0$,

where 0 is a propositional constant.

$9\ r/0\ {*C}12—13$,

13 $CCpNqCqNp$.

$11\ q/0\ {*C}12—14$,

14 $CCpNpNp$.

Therefore we have the NB-system.

If we add two axioms:

15 $CCqpCCCpqqp$

and Wajsberg axiom

16 $C0p$,

then as already shown in A. N. Prior ([4], p. 51), by these axioms we have

17 $CNNpp$.

Hence we have the Russell system of the classical propositional calculus.

We have some axiom systems of the positive implicational calculus, for example, the single axiom by C. A. Meredith [3]:

18 $CCCpqrCsCCqCrtCqt$

or

19 $CtCCpqCCCspCqrCpr$.

The proof of $18 \Rightarrow 7$, 8 is found in C. A. Meredith [3]. On the other hand, recently S. Tanaka gives a proof of $19 \Rightarrow 7$, 8. They have not given the proofs of these converses, so we give the proofs by an algebraic technique (for details, see [2]). Following my method, axioms 7, 8 are written in the forms of

20 $$p*q \leqslant p,$$

21 $$(r*p)*(q*p) \leqslant (r*q)*p.$$

Thesis 9 means a commutative law:

22 $$(r*p)*q \leqslant (r*q)*p.$$

As shown in [1], we have:

23 $p \leqslant q$ implies $p*r \leqslant q*r$ and $r*q \leqslant r*p$

in the positive implicational calculus. To prove thesis 19, consider $q*p \leqslant p$ and $(t*r)*(t*r)=0$, i.e. $t*(t*r) \leqslant r$ by (22). By (23), we have

$$(t*(t*r))*q \leqslant r*q \leqslant r*(q*p).$$

On the other hand, by (21), we have

$$(t*q)*((t*r)*q) \leqslant (t*(t*r))*q.$$

By these two results, then

$$(t*q)*((t*r)*q) \leqslant r*(q*p),$$

hence

$$((t*q)*((t*r)*q))*(r*(q*p)) \leqslant s.$$

By (22), we have

$$((t*q)*((t*r)*q))*s \leqslant r*(q*p),$$

which is the thesis 19.

To prove the thesis 20, consider $p*s \leqslant p$, then by (23), we have

$$(r*q)*p \leqslant (r*q)*(p*s).$$

By (21),

$$(r*p)*(q*p) \leqslant (r*q)*p,$$

hence

$$(r*p)*(q*p) \leqslant (r*q)*(p*s).$$

Then, by (22), we have

$$(r*p)*((r*q)*(p*s)) \leqslant q*p,$$

which means

$$((r*p)*((r*q)*(p*s)))*(q*p) \leqslant t.$$

This is the algebraic form of the thesis (20). We complete the proofs of (19), (20). Therefore we have the following

Theorem 1. *The NB-system is obtained from one of*

(1) *CpCqp, CCpCqrCCprCqr, Np=Cp*0,

(2) *CCCpqrCsCCqCrtCqt, Np=Cp*0,

(3) *CtCCpqCCCspCqrCpr, Np=Cp*0.

Further we have

Theorem 2. *The classical propositional calculus is characterized by one of*

(1) *CpCqp, CCpCqrCCpqCpr, CCqpCCCpqqp, C*0*p*,

(2) *CCCpqrCsCCqCrtCqt, CCqpCCCpqqp, C*0*p*,

(3) *CtCCpqCCCspCqrCpr, CCqpCCCpqqp, C*0*p*,

*where we define Np=Cp*0.

References

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