249. Note on the Representation of Semi-Groups of Non-Linear Operators

By Shinnosuke ÔHARU
Waseda University

(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1966)

1. Let X be a Banach space and let $\{T(\xi)\}_{\xi\geqq 0}$ be a family of non-linear operators from X into itself satisfying the following conditions:

(1) $T(0)=I,\ T(\xi)T(\eta)=T(\xi+\eta) \qquad \xi, \eta \geqq 0,$

(2) $\| T(\xi)x-T(\xi)y \| \leqq \| x-y \| \qquad \xi>0,\ x, y \in X,$

(3) There exists a dense subset D in X such that for each $x \in D$, the right derivative

$$D_\xi^+ T(\xi)x=\lim_{h\to 0+} h^{-1}(T(\xi+h)x-T(\xi)x)$$

exists and it is continuous for $\xi \geqq 0$. Then we shall call this family $\{T(\xi)\}_{\xi\geqq 0}$ a *non-linear contraction semi-group*.

Definition. *We define the infinitesimal generator A of a non-linear contraction semi-group $\{T(\xi)\}_{\xi\geqq 0}$ by*

$$Ax=\lim_{h\to 0+} A_h x$$

whenever the limit exists, where $A_h=h^{-1}(T(h)-I)$. We denote the domain of A by $D(A)$.

Lately J. W. Neuberger [1] gave the following result: If $\{T(\xi)\}_{\xi\geqq 0}$ is a non-linear contraction semi-group,*) then for each $x\in X$ and each $\xi\geqq 0$

$$\lim_{n\to\infty} \limsup_{\delta\to 0+} \| (I-(\xi/n)A_\delta)^{-n}x-T(\xi)x \| =0.$$

It is well known that if $\{T(\xi)\}_{\xi\geqq 0}$ is a linear contraction semi-group of class (C_0), then for each $x \in X$ and each $\xi \geqq 0$

$$\lim_{n\to\infty}(I-(\xi/n)A)^{-n}x=T(\xi)x$$

(see [2]). In this paper we shall give the representation of this type for non-linear contraction semi-groups.

The main results are the follwing

Theorem. *Let $\{T(\xi)\}_{\xi\geqq 0}$ be a non-linear contraction semi-group and let A be the infinitesimal generator such that $\overline{\Re(I-\xi_0 A)}=X$ for some $\xi_0>0$. Then for each $\xi>0$ there exists an inverse operator $(I-\xi A)^{-1}$ and its unique extension $L(\xi)$ onto X, which is a contraction operator, and $T(\xi)$ is represented by*

---

*) In his paper the following condition is assumed:

(3)′ There is a dense subset D of X such that if x is in D, then the derivative $T'(\xi)x$ is continuous with domain $[0, \infty)$.

$$\lim_{n\to\infty} L(\xi/n)^n x = T(\xi)x \qquad \xi \geqq 0,\ x \in X,$$

where for each fixed $x \in X$ *the convergence is uniform for any compact set in* $[0, \infty)$ *and for each fixed* $\xi \geqq 0$ *it is the continuous convergence on* X. *Moreover, there exists a unique mapping* $\tilde{A}$, *which is not necessarily one-valued, defined on a region* $\tilde{D} \supset D(A)$ *such that*

(1) *the mapping* $\tilde{D} \ni x \to x - \xi\tilde{A}x$ *is the topological inverse mapping of* $L(\xi)$,

(2) $\tilde{A}x \ni Ax$ *for each* $x \in D(A)$,

(3) *for any* $x \in \tilde{D}$ *there exists a sequence* $\{x_n\} \subset D(A)$ *such that* $\lim x_n = x$ *and* $\lim_{n\to\infty} Ax_n \in \tilde{A}x$.

Corollary 1. *If* $\tilde{A}$ *is one-valued, then in the above Theorem* $L(\xi) = (I - \xi\tilde{A})^{-1}$ *and* $\tilde{A}$ *is the closure of* A *in the sense that the graph* $G(\tilde{A})$ *of* $\tilde{A}$ *is the closure of the graph* $G(A)$ *in* $X \times X$.

Corollary 2. *If* $\Re(I - \xi_0 A) = X$ *for some* $\xi_0 > 0$, *then* $\tilde{A} = A$ *in the above Corollary* 1.

2. We shall prove the theorems mentioned above by the following successive lemmas:

Lemma 1. $D(A) \supset D$, $D(A) \supset T(\xi)[D]$ *for each* $\xi \geqq 0$. *And the left derivative also exists, and is equal to the right one and*

$$\frac{d}{d\xi} T(\xi)x = AT(\xi)x$$

on $(0, \infty)$ *for each* $x \in D$.

Proof. The first relations of inclusion follow immediately from the condition (3). It follows from

$$\| T(\xi \pm h)x - T(\xi)x \| \leqq \| T(h)x - x \| \qquad (x \in D)$$

and the denseness of D that for any $x \in X$, $T(\xi)x$ is strongly continuous on $[0, \infty)$. Therefore by the same argument as in the linear case we get the above conclusions (see [3]; p. 239). Q.E.D.

Under the conditions (1)-(3) and by virtue of Lemma 1, we can apply the Neuberger's results and get the following

Lemma 2. *For each* $\xi > 0$ *and* $\delta > 0$, $(I - \xi A_\delta)^{-1}$ *exists on* X *and is a contraction operator in the sense that*

$$\|(I - \xi A_\delta)^{-1}x - (I - \xi A_\delta)^{-1}y\| \leqq \| x - y \| \qquad x, y \in X.$$

Lemma 3. *For each* $\xi > 0$, $(I - \xi A)^{-1}$ *exists on* $\Re(I - \xi A)$ *and contraction operator there. And if* $\Re(I - \xi A)$ *is dense in* X, *then the family* $\{(I - \xi A_\delta)^{-1}\}_{\delta > 0}$ *of contraction operators converges to some contraction operator* $L(\xi)$ *defined on* X *onto some region* $\tilde{D}_\xi \supset D(A)$. *This* $L(\xi)$ *is a unique extension of* $(I - \xi A)^{-1}$.

Proof. Let $\tau(x, y)$ be defined by $\lim_{a\to 0+} a^{-1}\{\| x + ay \| - \| x \|\}$. This always exists for each $x, y \in X$ and has the following properties [4]:

(i) $|\tau(x, y)| \leqq \|y\|$,

(ii) $\tau(x, y+z) \leqq \tau(x, y)+\tau(x, z)$,

(iii) $\tau(x, \lambda x+cy)=\Re_e(\lambda)\|x\|+c\tau(x, y) \qquad (c \geqq 0)$.

Using these properties, for any $u, v \in D(A)$ and $\delta>0$ we have

$$\begin{aligned}\tau(u-v, A_\delta u-A_\delta v)&=\tau\Big(u-v, \frac{T(\delta)u-T(\delta)v}{\delta}-\frac{u-v}{\delta}\Big)\\ &\leqq \tau(u-v, \delta^{-1}(T(\delta)u-T(\delta)v)-\delta^{-1}\|u-v\|\\ &\leqq \delta^{-1}\{\|T(\delta)u-T(\delta)v\|-\|u-v\|\}\leqq 0.\end{aligned}$$

From the continuity of $\tau(u-v, \cdot)$ we have $\tau(u-v, Au-Av)\leqq 0$ for each $u, v \in D(A)$. Thus we have again from (i), (ii), and (iii) the following estimate for any $u, v \in D(A)$:

$$\begin{aligned}\|(I-\xi A)u-(I-\xi A)v\| &\geqq \tau(u-v, (u-v)-\xi(Au-Av))\\ &\geqq \|u-v\|-\xi\tau(u-v, Au-Av)\geqq\|u-v\|,\end{aligned}$$

which implies the first assertion. For any $x \in \Re(I-\xi A)$ we have, from Lemma 2,

$$\begin{aligned}&\|(I-\xi A_\delta)^{-1}x-(I-\xi A)^{-1}x\|\\ &\leqq \|(I-\xi A)(I-\xi A)^{-1}x-(I-\xi A_\delta)(I-\xi A)^{-1}x\|\\ &=\xi\,\|A_\delta(I-\xi A)^{-1}x-A(I-\xi A)^{-1}x\|\to 0 \text{ as } \delta\to 0.\end{aligned}$$

Thus we have

$$\lim_{\delta\to 0+}(I-\xi A_\delta)^{-1}x=(I-\xi A)^{-1}x \qquad (*)$$

for any $x \in \Re(I-\xi A)$. On the other hand, each $(I-\xi A_\delta)^{-1}$ is a contraction operater defined on X from Lemma 2, and so, combining with (*) and the denseness of $\Re(I-\xi A)$, it follows that the family $\{(I-\xi A_\delta)^{-1}\}_{\delta>0}$ converges to some contraction operator $L(\xi)$ defined on X and that this $L(\xi)$ is the unique extension of $(I-\xi A)^{-1}$. Q.E.D.

Lemma 4. *If* $\overline{\Re(I-\xi_0 A)}=X$ *for some* $\xi_0>0$, *then* $\overline{\Re(I-\xi A)}=X$ *for any* $\xi>0$. *And if* $\Re(I-\xi_0 A)=X$ *for some* $\xi_0>0$, *then* $\Re(I-\xi A)=X$ *for any* $\xi>0$.

Proof. Since $\overline{\Re(I-\xi_0 A)}=X$, from Lemma 3 there exists a unique extension $L(\xi_0)$ of $(I-\xi_0 A)^{-1}$, which is also a contraction. Changing $I-\xi A$ to the form

$$I-\xi A=\frac{\xi}{\xi_0}\Big[I-\Big(1-\frac{\xi_0}{\xi}\Big)L(\xi_0)\Big](I-\xi_0 A);$$

for any $x \in X$, we put $Ky=x+(1-(\xi_0/\xi))L(\xi_0)y$ for each $y \in X$. Then K becomes a contraction mapping for ξ with $(\xi_0/2)<\xi$, since $\|Ky-Ky'\|\leqq|1-(\xi_0/\xi)|\cdot\|y-y'\|$. Thus there exists a unique fixed point z of K; $Kz=z$, and so we have

$$x=z-(1-(\xi_0/\xi))L(\xi_0)z=[1-(1-(\xi_0/\xi))L(\xi_0)]z.$$

Since $\overline{\Re(I-\xi_0 A)}=X$, there exists a sequence $\{x_n\}\subset\Re(I-\xi_0 A)$ such that $\lim_{n\to\infty}x_n=z$. Putting $y_n=(I-\xi_0 A)^{-1}x_n$, we have

$$\frac{\xi}{\xi_0}\Big[I-\Big(1-\frac{\xi_0}{\xi}\Big)L(\xi_0)\Big]x_n=\frac{\xi}{\xi_0}\Big[I-\Big(1-\frac{\xi_0}{\xi}\Big)L(\xi_0)\Big](I-\xi_0 A)y_n=(I-\xi A)y_n,$$

where the left hand side tends to $(\xi/\xi_0)x$ as $n\to\infty$. Therefore it follows that $\overline{\Re(I-\xi A)}=X$ for all $\xi>(\xi_0/2)$. Thus in particular $\overline{\Re(I-(2\xi_0/3)A)}=X$. Again change the $I-\xi A$ to the form

$$I-\xi A=(3\xi/2\xi_0)[I-(1-(2\xi_0/3\xi))L(2\xi_0/3)](I-(2\xi_0/3)A).$$

For any $x\in X$, putting $K_1y=x+(1-(2\xi_0/3\xi))L(2\xi_0/3)y$ for each $y\in X$, K_1 becomes a contraction mapping for ξ with $(\xi_0/3)<\xi$. In the similar way as in the abovementioned we have $\overline{\Re(I-\xi A)}=X$ for $\xi>(\xi_0/3)$. Inductively we can prove $\overline{\Re(I-(\xi_0/k)A)}=X$ $(k=3, 4, 5, \cdots)$ and thus we have $\overline{\Re(I-\xi A)}=X$ for $\xi>0$. The last assertion is now evident. Q.E.D.

By virtue of this Lemma 4, we assume in the following Lemmas that $\overline{\Re(I-\xi_0 A)}=X$ for some $\xi_0>0$, which insures the existence of the limit operator $L(\xi)$ for each $\xi>0$ (by Lemma 3).

Lemma 5. *The relation*

$$L(\xi)\left[\frac{\xi}{\xi'}y+\frac{\xi'-\xi}{\xi'}L(\xi')y\right]=L(\xi')y$$

holds for any $y\in X$ and $\xi,\xi'>0$. And $L(\xi)[X]=L(\xi')[X]$ for any $\xi,\xi'>0$. In particular, $\tilde{D}_\xi$ of Lemma 3 is independent of $\xi>0$.

Proof. For any $\delta>0$, $\xi,\xi'>0$ and $y\in X$, we have

$$(I-\xi'A_\delta)^{-1}y=(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}(I-\xi'A_\delta)^{-1}y+\frac{\xi}{\xi'}y\right]$$

and thus

$$\left\|L(\xi')y-L(\xi)\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]\right\|\leqq\|L(\xi')y-(I-\xi'A_\delta)^{-1}y\|$$

$$+\left\|(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}(I-\xi'A_\delta)^{-1}y+\frac{\xi}{\xi'}y\right]-(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]\right\|$$

$$+\left\|(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]-L(\xi)\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]\right\|.$$

Passing to the limit as $\delta\to0$, we have the required relation for each $y\in X$. From this it follows that $L(\xi')[X]\subset L(\xi)[X]$ for any $\xi,\xi'>0$ and thus we have $L(\xi')[X]=L(\xi)[X]$. The last assertion is now evident. Q.E.D.

By virtue of this Lemma, we denote the set $L(\xi)[X]=\tilde{D}_\xi$, independent of $\xi>0$, by $\tilde{D}$.

Lemma 6. *For any $\xi,\xi'>0$ we have the relation of inclusion: $\frac{1}{\xi}(x-L(\xi)^{-1}x)=\frac{1}{\xi'}(x-L(\xi')^{-1}x)\subset X$, $x\in\tilde{D}$, where $L(\xi)^{-1}$ is the topological inverse mapping of $L(\xi)$.*

Proof. It suffices to prove that for any $x\in\tilde{D}$, $\xi,\xi'>0$

$$\xi^{-1}(x-L(\xi)^{-1}x)\supseteqq\xi'^{-1}(x-L(\xi')^{-1}x).$$

From Lemma 5, $L(\xi)^{-1}L(\xi)\left[\frac{\xi}{\xi'}y+\frac{\xi'-\xi}{\xi'}L(\xi')y\right]=L(\xi)^{-1}L(\xi')y$. Thus

$L(\xi)^{-1}L(\xi')y \ni \frac{\xi}{\xi'}y + \frac{\xi'-\xi}{\xi'}L(\xi')y$ for each $y \in X$. Therefore we have $L(\xi')y - y \in (\xi'/\xi)L(\xi')y - (\xi'/\xi)L(\xi)^{-1}L(\xi')y$ for each $y \in X$. And for any $u \in L(\xi')^{-1}x$ we have

$$\xi^{-1}(x - L(\xi)^{-1}x) = \xi^{-1}(L(\xi')u - L(\xi)^{-1}L(\xi')u)$$
$$= \xi'^{-1}((\xi'/\xi)L(\xi')u - (\xi'/\xi)L(\xi)^{-1}L(\xi')u).$$

From this and the abovementioned it follows that the above right hand side contains the element $\xi'^{-1}(L(\xi')u - u)$, which implies the required relation of inclusion. Q.E.D.

Lemma 7. *The not necessarily one valued mapping* $\tilde{A}$ *is defined on* $\tilde{D}$ *by*

$$\tilde{A}x = \xi^{-1}(x - L(\xi)^{-1}x) \subset X \qquad x \in \tilde{D},$$

which has the properties (1)–(3) *mentioned in the main theorem.*

Proof. Such an operator $\tilde{A}$ is well defined by Lemma 6. For each $x \in \tilde{D}$ we have

$$\xi\tilde{A}x = x - L(\xi)^{-1}x \subset X \text{ and so, } x - \xi\tilde{A}x = L(\xi)^{-1}x \subset X.$$

But since $L(\xi)[x - \xi\tilde{A}x] = L(\xi)[L(\xi)^{-1}x] = x$, the mapping $x \to x - \xi\tilde{A}x$ is the topological inverse mapping of $L(\xi)$, which implies (1). Since $L(\xi)$ is the unique extension of $(I - \xi A)^{-1}$ by Lemma 3, $L(\xi)(I - \xi A)x = x$ for each $x \in D(A)$ and thus $L(\xi)^{-1}x = x - \xi\tilde{A}x \ni (I - \xi A)x$, from which $\tilde{A}x \ni Ax$. Thus (2) is proved. Finally we shall prove (3). For any $x \in \tilde{D}$ there exists $x' \in X$ such that $x = L(1)x'$. Since $\Re(I - A)$ is dense in X, there exists a sequence $\{x_n\} \subset D(A)$ such that $(I - A)x_n \to x'$ as $n \to \infty$. Thus $x_n = L(1)(I - A)x_n \to L(1)x' = x$ and so, $Ax_n = x_n - (I - A)x_n \to x - x' \in x - L(1)^{-1}x = \tilde{A}x$. Q.E.D.

Lemma 8. *For each* $\xi \geqq 0$, $\{L(\xi/n)^n\}$ *converges continuously to* $T(\xi)$ *on* X *and for each* $x \in X$, $\{L(\xi/n)^n x\}$ *converges to* $T(\xi)x$ *uniformly in* ξ *for any compact subset in* $[0, \infty)$.

Proof. Since $T'(\xi)x = AT(\xi)x$ for each $x \in D$ from Lemma 1, we have the following estimate:

$$
\begin{aligned}
&\| L(\xi/n)^n x - T(\xi)x \| \\
&\quad = \| L(\xi/n)^n x - T(\xi/n)^n x \| \\
&\quad \leqq \sum_{i=1}^{n} \| L(\xi/n)^{n-i+1} T(\xi(i-1)/n)x - L(\xi/n)^{n-i} T(\xi i/n)x \| \\
&\quad \leqq \sum_{i=1}^{n} \| L(\xi/n) T(\xi(i-1)/n)x - L(\xi/n)(I - (\xi/n)A) T(\xi i/n)x \| \\
&\quad \leqq \sum_{i=1}^{n} (\xi/n) \| A_{\frac{\xi}{n}} T(\xi(i-1)/n)x - AT(\xi i/n)x \| \\
&\quad = \sum_{i=1}^{n} (\xi/n) \| (\xi/n)^{-1}(T(\xi i/n)) - T(\xi(i-1)/n)x - AT(\xi i/n)x \| \\
&\quad \leqq (\xi/n) \sum_{i=1}^{n} (\xi/n)^{-1} \int_{\frac{\xi(i-1)}{n}}^{\frac{\xi i}{n}} \| T'(\sigma)x - T'(\xi i/n)x \| d\sigma \\
&\quad \leqq \xi \max_{1 \leqq i \leqq n} \max_{\sigma \in \left[\frac{\xi(i-1)}{n}, \frac{\xi i}{n}\right]} \| T'(\sigma)x - T'(\xi i/n)x \|.
\end{aligned}
$$

The above right hand tends to 0 as $n \to \infty$, since $T'(\sigma)x$ is uniformly continuous on $[0, \xi]$. Thus $\lim_{n\to\infty} L(\xi/n)^n x = T(\xi)x$ for each $x \in D$. On the other hand, $L(\xi/n)^n$ is a contraction operator for each n. And so, $\{L(\xi/n)^n\}$ converges continuously to $T(\xi)$ on X [5]. Moreover the uniform convergence in ξ for any compact subset of $[0, \infty)$ is evident from the abovementioned estimate. Q.E.D.

Finally the author expresses his sincere thanks to Professor I. Miyadera for many useful advices.

References

[1] J. W. Neuberger: An exponential formula for one-parameter semi-groups of non-linear transformations. J. Math. Soc. Japan, **18**(**2**), 154–157, (1966).
[2] E. Hille and R. S. Phillips: Functional analysis and semi-groups. American Math. Soc. (1957).
[3] K. Yosida: Functional Analysis. Berlin-Heidelberg-New York: Springer (1965).
[4] M. Hasegawa: On contraction semi-groups and (di)-operators. J. Math. Soc. Japan, **18**(**3**), 303–330 (1966).
[5] W. Rinow: Die Innere Geometrie der Metrischen Räume. Berlin-Göttingen-Heidelberg: Springer (1961).