243. On Julia's Exceptional Functions

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1. Let $f(z)$ be a transcendental meromorphic function and $\rho(f(z))$ the spherical derivative of $f(z)$. O. Lehto and K. I. Virtanen ([2]) have proved that $f(z)$ satisfies

$$(*) \qquad \rho(f(z))=O(1/|z|) \qquad (z\to\infty)$$

if and only if $f(z)$ is a Julia's exceptional function.

Recently, J. M. Anderson and J. Clunie ([1]) have raised an open question whether a function satisfying the condition $(*)$ can possess a Valiron deficient value. We give a negative answer for it in this paper.

2. Let $f(z)$ be a transcendental meromorphic function and $\{\sigma_\nu\}$ an arbitrary sequence of complex numbers such that $|\sigma_\nu|\to\infty$ as $\nu\to\infty$ and $|\sigma_\nu|\geqq 1$. We put $f_\nu(z)=f(\sigma_\nu z)$. If, for every such $\{\sigma_\nu\}$, the family $\{f_\nu(z)\}$ is normal in the sense of Montel in $0<|z|<\infty$, $f(z)$ is said to be a Julia's exceptional function (c.f. A. Ostrowski [4]).

We shall assume the acquaintance with the standard terminology of the Nevanlinna theory:

$$T(r,f)\ , \qquad n(r,a,f)\ , \qquad N(r,a,f)\ , \qquad m(r,a,f)$$

and with the first fundamental theorem of R. Nevanlinna ([3]). The deficiencies of Nevanlinna $\underline{\delta}(a,f)$ and of Valiron $\bar{\delta}(a,f)$ of a value a are defined respectively as follows:

$$\underline{\delta}(a,f)=\varliminf_{r\to\infty}\frac{m(r,a,f)}{T(r,f)}$$

and

$$\bar{\delta}(a,f)=\varlimsup_{r\to\infty}\frac{m(r,a,f)}{T(r,f)}\ .$$

If $\underline{\delta}(a,f)>0$ $(\bar{\delta}(a,f)>0)$, the value a is said to be a Nevanlinna (Valiron) deficient value.

Theorem. *If $f(z)$ is a Julia's exceptional function, then for every complex number a $\bar{\delta}(a,f)=0$.*

Proof. A. Ostrowski ([4]) has proved that if $f(z)$ is a Julia's exceptional function, there exists a finite number C independent of r such that

$$|n(r,0,f)-n(r,\infty,f)|<C.$$

Hence we have

$$n(r,\infty,f)-C<n(r,0,f)<n(r,\infty,f)+C,$$

so that it follows

$$N(r, \infty, f) - O(\log r) \leq N(r, 0, f) \leq N(r, \infty, f) + O(\log r).$$

Since $\lim_{r\to\infty} \log r / T(r, f) = 0$, it holds

$$(1) \qquad \varlimsup_{r\to\infty} \frac{N(r, \infty, f)}{T(r, f)} = \varlimsup_{r\to\infty} \frac{N(r, 0, f)}{T(r, f)}$$

From (1) and Nevanlinna's first fundamental theorem, we have immediately

$$(2) \qquad \bar{\delta}(0, f) = \bar{\delta}(\infty, f).$$

It is well-known that for any complex value a

$$(3) \qquad T(r, f) = T(r, f-a) + O(1)$$

and if $f(z)$ is a Julia's exceptional function, then so is $f(z)-a$.

From (2), (3) and the above fact, we have

$$\bar{\delta}(0, f) = \bar{\delta}(\infty, f) = \bar{\delta}(\infty, f-a) = \bar{\delta}(0, f-a) = \bar{\delta}(a, f)$$

for every finite complex value a. This implies that $\bar{\delta}(a, f)$ is independent of a, finite or not. On the other hand, the logarithmic capacity of the set of the Valiron deficient values is zero (c.f. R. Nevanlinna [3]), so that the deficiency of Valiron $\bar{\delta}(a, f)$ must be zero for any a, finite or not.

Remark. As an immediate corollary, we obtain Theorem 5 of J. M. Anderson and J. Clunie ([1]).

References

[1] J. M. Anderson and J. Clunie: Slowly growing meromorphic functions. Comm. Math. Helv., **40**, 267-280 (1966).

[2] O. Lehto and K. I. Virtanen: On the behaviour of meromorphic functions in the neighborhood of an isolated singularity. Ann. Acad. Sci. Fenn., **240** (1957).

[3] R. Nevanlinna: Eindeutige analytische Funktionen. Springer, Berlin 2. Aufl. (1953).

[4] A. Ostrowski: Über Folgen analytischen Funktionen und einige Verschärfungen des Picardschen Satzes. Math. Zeit., **24**, 215-258 (1925).