141. On Goursat Problem. I

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1. We shall consider the problem of the unique existence of the solutions in some Gevrey class for the equation written in the following form on $\Omega=\prod_{i=1}^{m}[0, T_i]\times D$ where D is the closure of a bounded domain, in $m+n$ dimensional euclidean space $\prod_{i=1}^{m} R^1_{t_i}\times R^n_x$, i.e. Goursat problem:

$$(1) \qquad \left(\frac{\partial}{\partial t}\right)^{\alpha} u(t, x)=\sum_{\beta,\gamma} a_{\beta\gamma}(t, x)\left(\frac{\partial}{\partial t}\right)^{\beta}\left(\frac{\partial}{\partial x}\right)^{\gamma} u(t, x)+f(t, x)$$

with data

$$(2) \qquad \left(\frac{\partial}{\partial t_i}\right)^{k} u(t, x)\,|_{t_i=0}=\phi_{ik}(t, x) \qquad 0\leqq k\leqq \alpha_i-1 \qquad 1\leqq i\leqq m,$$

where $\phi_{ik}(t, x)$ are defined on $t_i=0$ satisfying

$$(3) \qquad \left(\frac{\partial}{\partial t_i}\right)^{k} \phi_{jl}(t, x)\Big|_{t_i=0}=\left(\frac{\partial}{\partial t_j}\right)^{l} \phi_{ik}(t, x)\Big|_{t_j=0} \qquad i\neq j,\ 1\leqq i, j\leqq m,$$

the notations contained in the above mean

$(t, x)=(t_1, \cdots, t_m, x_1, \cdots, x_n)$,

$\alpha=(\alpha_1, \cdots, \alpha_m)$ multi-positive-integer,

$\beta=(\beta_1, \cdots, \beta_n)$, $\gamma=(\gamma_1, \cdots, \gamma_n)$ multi-nonnegative-integers,

$$\left(\frac{\partial}{\partial t}\right)^{\alpha}=\left(\frac{\partial}{\partial t_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial t_m}\right)^{\alpha_m}, \qquad \left(\frac{\partial}{\partial x}\right)^{\gamma}=\left(\frac{\partial}{\partial x_1}\right)^{\gamma_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\gamma_n},$$

and the summation $\sum_{\beta,\gamma}$ is done for all β, γ satisfying

$$(4) \qquad |\alpha|\geqq|\beta|+|\gamma|, \qquad |\alpha|>|\beta| \text{ and } \alpha_i\geqq\beta_i \qquad 1\leqq i\leqq m,$$

where $|\alpha|=\sum_{i=1}^{m}\alpha_i$ and $|\beta|, |\gamma|$ are similarly defined.

A. Friedman solved the equation with non-linear right hand side under the assumption of the analyticity with respect to t_i variables on $a_{\beta\gamma}(t, x)$ and $f(t, x)$ and a rather stronger condition than (4), [1]. It seems for me that this assumption on t_i variables is essential in his proofs even when we restrict the equation in the linear case. The purpose of this note is to give a remark that we can get a similar result for the linear case under the assumption of the continuity with respect to t_i variables. On this problem Darboux, Goursat, and Bendom treated the case for $m=2, \alpha_1=\alpha_2=1$ and a non-linear right hand side, [2]. L. Hörmander solved the case for analytic $a_{\beta\gamma}(t, x)$ and $f(t, x)$ under a weaker condition than

(4), [3]. As for the method we make a extended use of that of C. Pucci, de Giorgi, and G. Talenti who used for the Cauchy problem, [4]—[7]. We only give the sketch of proofs, and the details will be published with further results on non-linear equations.

2. A function $g(t, x)$ in $C^{(0,\infty)}_{(tx)}(\Omega)$[1] is defined to be in the class $G(\delta)$ $\delta \geqq 1$ *if the estimate*

$$\max_{\Omega} \left| \left(\frac{\partial}{\partial x}\right)^{s} g(t, x) \right| \leqq \rho^{|s|+1} \Gamma(\delta\,|\,s\,|+1)^{2)}$$

holds for some constant ρ and any multi-integer s. Here we state

Theorem 1 (*existence*).

(I) *$f(t, x)$, $a_{\beta\gamma}(t, x)$, and $\phi_{ik}(t, x)$ are in $G(\delta)$ for δ:*

$$1 \leqq \delta \leqq \min_{\beta,\gamma} \frac{|\alpha| - |\beta|}{|\gamma|}$$

(II) *For any ε: $0 < \varepsilon < 1$ fixed, we choose $T = (T_1, \cdots, T_m)$[3] such that*

$$A(T) = \sum_{\beta,\gamma} \left\{ \frac{T}{\varepsilon^{m\delta}(1-\varepsilon)} \right\}^{\alpha-\beta\,4)} < 1 \tag{5}$$

holds. Then there is a solution $u(t, x)$ of (1), (2) *satisfying*

$$\left(\frac{\partial}{\partial t}\right)^{\alpha} u(t, x) \in G(\delta).$$

Theorem 2 (*uniqueness*). *If $u_1(t, x)$, $u_2(t, x)$ are two solutions of* (1), (2) *satisfying $u_i(t, x) \in G(\delta)$ $i = 1, 2$, and $a_{\beta\gamma}(t, x)$ are in $G(\delta)$, then $u_1(t, x) \equiv u_2(t, x)$ on Ω.*

We can set without loosing generality $\phi_{ik}(t, x) \equiv 0$ by replacing $u(t, x)$ in (1) for

$$v(t, x) = u(t, x) - \sum_{i=1}^{m} \sum_{k=0}^{\alpha_i - 1} \frac{t_i^k}{k!} \phi_{ik}(t, x) + \sum_{i<j} \sum_{k=0}^{\alpha_i - 1} \sum_{l=0}^{\alpha_j - 1} \frac{t_i^k}{k!} \frac{t_j^l}{l!} \phi_{jl}(t, x)\,|_{t_j=0},$$

and the new $f(t, x)$ in the resulting form of (1) by this replacement is denoted by $g(t, x)$, also in $G(\delta)$ by (I) of theorem 1.

Setting

$$w(t, x) = \left(\frac{\partial}{\partial t}\right)^{\alpha} v(t, x)$$

we have

$$v(t, x) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} w(\tau, x) d\tau$$

where $t \in \prod_{i=1}^{m} [0, T_i]$,

1) $C^{(0,\infty)}_{(t,x)}(\Omega)$ is the totality of the functions continuous with respect to t and infinitely differentiable with respect to x on Ω.

2) $\Gamma(a)$ is the Gamma function.

3) Ω depends on $T = (T_1, \cdots, T_m)$.

4) For $T = (T_1, \cdots, T_m)$ and $\mu = (\mu_1, \cdots, \mu_m)$, $T^{\mu} = T_1^{\mu_1} \cdots T_m^{\mu_m}$.

$$\int_0^t = \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_m}, \qquad (\alpha-1)! = (\alpha_1-1)!\,(\alpha_2-1)!\cdots(\alpha_m-1)!,^{5)}$$

$(t-\tau)^{\alpha-1} = (t_1-\tau_1)^{\alpha_1-1}(t_2-\tau_2)^{\alpha_2-1}\cdots(t_m-\tau_m)^{\alpha_m-1}$ and $d\tau = d\tau_1 d\tau_2\cdots d\tau_m$. By this we can transform (1), (2) into the following integro-differential equation

$$(6) \qquad w(t,x) = g(t,x) + Hw(t,x)$$

$$H = \sum_{\beta\,\gamma} H_{\beta\gamma}, \qquad Hw(t,x) = a_{\beta\gamma}(t,x)\int_0^t \frac{(t-\tau)^{\alpha-\beta-1}}{(\alpha-\beta-1)!}\left(\frac{\partial}{\partial x}\right)^{\gamma} w(\tau,x)\,d\tau$$

By calculating of Gamma function we can obtain

Lemma 1. *If $a_{\beta\gamma}(t,x)$ is in $G(\delta)$ and $b(t,x)$ in $C^{(0,\infty)}_{(t,x)}(\Omega)$ satisfies*

$$\left|\left(\frac{\partial}{\partial x}\right)^{s} b(t,x)\right| \leqq B\rho^{|s|}\Gamma(\delta\,|s|+l+1)$$

then

$$\left|\left(\frac{\partial}{\partial x}\right)^{s}\{a_{\beta\gamma}(t,x)b(t,x)\}\right| \leqq \frac{B\rho}{l+1}\rho^{|s|}\Gamma(\delta\,|s|+l+2)$$

holds.

Lemma 2. *For multi-integers (β^i,γ^i) $1\leqq i\leqq p$ satisfying* (4)

$$(7) \qquad \left|\left(\frac{\partial}{\partial x}\right)^{s} H_{\beta^p\gamma^p}H_{\beta^{p-1}\gamma^{p-1}}\cdots H_{\beta^1\gamma^1}g(t,x)\right| \leqq \frac{\rho^{\sum_{i=1}^{p}|\gamma^i|+|s|+p+1}}{\left\{\sum_{i=1}^{p}(\alpha-\beta^i)\right\}!\,p!}\,t^{\sum_{i=1}^{p}(\alpha-\beta^i)}$$

holds for $g(t,x)$ in (6).

This lemma is proved as the following. We set $h(t,x) = H_{\beta^{p-1}\gamma^{p-1}}\cdots H_{\beta^1\gamma^1}g(t,x)$ satisfying (7) for $p-1$. Then by lemma 1

$$\left|\left(\frac{\partial}{\partial x}\right)^{s} H_{\beta^p\gamma^p}h(t,x)\right| = \left|\left(\frac{\partial}{\partial x}\right)^{s} a_{\beta^p\gamma^p}(t,x)\int_0^t \frac{(t-\tau)^{\alpha-\beta^p-1}}{(\alpha-\beta^p-1)!}\left(\frac{\partial}{\partial x}\right)^{\gamma^p} h(\tau,x)\,d\tau\right|$$

$$\leqq \frac{\rho^{\sum_{i=1}^{p-1}|\gamma^i|+|s|+(p-1)+1}}{\left\{\sum_{i=1}^{p-1}(\alpha-\beta^i)\right\}!\,(p-1)!}\;\frac{\rho}{\delta\sum_{i=1}^{p}|\gamma^i|+\delta\,|s|+p+1}$$

$$\times\Gamma\left(\delta\sum_{i=1}^{p}|\gamma^i|+\delta\,|s|+p+1\right)\cdot\frac{1}{(\alpha-\beta^p-1)!}$$

$$\times\int_0^t \tau^{\sum_{i=1}^{p-1}(\alpha-\beta^i)}(t-\tau)^{\alpha-\beta^p-1}\,d\tau$$

is obtained. And by

$$\int_0^t \tau^{\sum_{i=1}^{p-1}(\alpha-\beta^i)}(t-\tau)^{\alpha-\beta^p-1}\,d\tau = t^{\sum_{i=1}^{p}(\alpha-\beta^i)}\int_0^t \tau^{\sum_{i=1}^{p-1}(\alpha-\beta^i)}(1-\tau)^{\alpha-\beta^p-1}\,d\tau$$

$$= t^{\sum_{i=1}^{p}(\alpha-\beta^i)}\frac{\left\{\sum_{i=1}^{p-1}(\alpha-\beta^i)\right\}(\alpha-\beta^p-1)!}{\left\{\sum_{i=1}^{p}(\alpha-\beta^i)\right\}!},$$

we get (7). The repeated use of the inequality

5) In what follows for $\mu=(\mu_1,\cdots,\mu_m)$, $\mu! = \mu_1!\cdots\mu_m!$

$$\Gamma(a+b+1) \leqq \frac{\Gamma(a+1)\Gamma(b+1)}{\varepsilon^a(1-\varepsilon)^b}$$

for any $a \geqq 0, b \geqq 0$, and any $\varepsilon : 0<\varepsilon<1$ fixed, and (4) lead to

$$\Gamma\Big(\delta \sum_{i=1}^{p} |\gamma^i| + \delta |s| + p + 1\Big) \leqq \Gamma\Big(\sum_{j=1}^{p} (|\alpha| - |\beta^i|) + \delta |s| + 1\Big)$$

$$\leqq \frac{\Gamma\Big(\sum_{i=1}^{p} (|\alpha| - |\beta^i|) + 1\Big) \Gamma(\delta |s| + p + 1)}{\varepsilon^{m\delta \sum_{i=1}^{p} |\gamma^i|} (1-\varepsilon)^{\delta |s| + p}}$$

$$\leqq \frac{\Big\{\sum_{i=1}^{p} (\alpha - \beta^i)\Big\}! \, \Gamma(\delta |s| + p + 1)}{\varepsilon^{m\delta \sum_{i=1}^{p} |\gamma^i|} (1-\varepsilon)^{\delta |s| + p} \varepsilon^{m\delta \sum_{i=1}^{p} (|\alpha| - |\beta^i|)} (1-\varepsilon)^{\sum_{i=1}^{p} (|\alpha| - |\beta^i|)}}.$$

Thus we get

$$\Big|\Big(\frac{\partial}{\partial x}\Big)^s H_{\beta^p \gamma^p} \cdots H_{\beta^1 \gamma^1} g(t, x)\Big| \leqq (1-\varepsilon)^\delta \Big\{\frac{\rho}{(1-\varepsilon)^\delta}\Big\}^{|s|+1}$$

$$\times \Big(\frac{\rho}{\varepsilon^{m\delta}}\Big)^{\sum_{i=1}^{p} |\gamma^i|} \Big(\frac{\rho}{1-\varepsilon}\Big)^p \frac{1}{p!} \Gamma(\delta |s| + p + 1).$$

Using this inequality we get

$$(8) \quad \Big|\Big(\frac{\partial}{\partial x}\Big)^s H^p g(t, x)\Big| = \Big|\Big(\frac{\partial}{\partial x}\Big)^s \Big(\sum_{\beta, \gamma} H_{\beta\gamma}\Big)^p g(t, x)\Big|$$

$$\leqq (1-\varepsilon)^\delta \Big\{\frac{\rho}{(1-\varepsilon)^\delta}\Big\}^{|s|+1} \Big[\sum_{\beta, \gamma} \Big\{\frac{t}{\varepsilon^{m\delta}(1-\varepsilon)}\Big\}^{\alpha-\beta} \Big(\frac{\rho}{\varepsilon^{m\delta}}\Big)^{|\delta|} \frac{\rho}{1-\varepsilon}\Big]^p$$

$$\times \frac{1}{p!} \Gamma(\delta |s| + p + 1).$$

By (4) for any $\varepsilon : 0<\varepsilon<1$ fixed we can choose $T=(T_1, \cdots, T_m)$ such that (5) holds, therefore using

$$\sum_{p=0}^{\infty} \frac{\Gamma(a+p+1)}{p!} Z^p = \frac{\Gamma(a+1)}{(1-Z)^{a+1}} \qquad \text{for} \quad a \geqq 0, |Z|<1,$$

the series $\sum_{p=0}^{\infty} \Big(\frac{\partial}{\partial x}\Big)^s H^p g(t, x)$ converges uniformly on Ω, and

$$\Big|\sum_{p=0}^{\infty} \Big(\frac{\partial}{\partial x}\Big)^s H^p g(t, x)\Big| \leqq (1-\varepsilon)^\delta (1-A(T))^{\delta |s| + 1} \Big\{\frac{p}{(1-\varepsilon)^\delta}\Big\}^{|s|+1} \Gamma(\delta |s| + 1)$$

holds. This shows the function $w(t, x) = \sum_{p=0}^{\infty} H^p g(t, x)$ is in $G(\delta)$. Thus a solution of (6) is obtained.

For the uniqueness if $u_i(t, x)$ $i=1, 2$ are two solution of (1), (2) stated in Theorem 2, we set $w(t, x) = \Big(\frac{\partial}{\partial t}\Big)^\alpha \{u_1(t_1 x) - u_2(t, x)\}$ belonging to $G(\delta)$. As the $w(t, x)$ satisfies $w(t, x) = Hw(t, x)$, $w(t, x) = H^p w(t, x)$ for any $p>1$. Furthemore the inequality (8) shows $w(t, x) \leqq (1-\varepsilon)^\delta \{A(T)\}^p$ where $A(T)<1$. Thus, $w(t, x) \equiv 0$ on Ω.

References

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