129. *Locally Symmetric K-Contact Riemannian Manifolds*

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§ 1. **Introduction.** Let M be a contact Riemannian manifold with a contact form η, the associated vector field ξ, a (1, 1)-tensor ϕ and the associated Riemannian metric g:

$$\phi^i_j\xi^j=0, \qquad \eta_i\phi^i_j=0, \qquad \eta_i\xi^i=1,$$
$$\phi^i_j\phi^j_k=-\delta^i_k+\xi^i\eta_k,$$
$$g_{ij}\xi^j=\eta_i,$$
$$g_{ij}\phi^i_r\phi^j_s=g_{rs}-\eta_r\eta_s.$$

If ξ is a Killing vector field with respect to g, M is called a K-contact Riemannian manifold, and then ξ is an infinitesimal automorphism of this structure. And we have

(1.1) $$\nabla_j\xi^i=-\phi^i_j,$$

(1.2) $$R_{jk}\xi^k=(m-1)\eta_j, \qquad m=\dim M,$$

(1.3) $$\eta_r R^r{}_{jks}\xi^s=g_{jk}-\eta_j\eta_k.$$

Further if the following relation

(1.4) $$\eta_r R^r{}_{jkl}=g_{jk}\eta_l-g_{jl}\eta_k$$

is satisfied, M is called a Sasakian manifold ([2]).

§ 2. **Statement of results.** M is said locally symmetric if we have $\nabla_h R^i{}_{jkl}=0$. In this paper first in § 3 we prove the following

Theorem 1. *Any locally symmetric K-contact Riemannian manifold is Sasakian and has constant curvature* 1.

As an immediate consequence we have

Theorem 2. *Any complete, simply connected and locally symmetric K-contact Riemannian manifold is globally isometric with a unit sphere.*

Any compact semi-simple Lie group G has the positive definite Riemannian metric defined by the Killing form, which is locally symmetric and is not of constant curvature, provided dim $G>3$ (cf. [4], p. 122). Therefore we get

Corollary 3. *A compact semi-simple Lie group* G (dim $G>3$) *with the usual metric can not be a K-contact Riemannian manifold.*

Remark 4. With regard to this Corollary in [1] (p. 729) it was shown that, if G is a semi-simple Lie group on which a left G-invariant contact form η is defined, then G is 3-dimensional and locally isomorphic with either $O(3)$ or $SL(2)$.

It is known that the homogeneous holonomy group of a Sasakian manifold is the full special orthogonal group ([6]). As for a K-contact Riemannian manifold, in § 4, we prove the following

Proposition 5. *The restricted homogeneous holonomy group of a K-contact Riemannian manifold is irreducible.*

Then we have ([5])

Corollary 6. *If the Ricci tensor field is parallel in a K-contact Riemannian manifold, then it is an Einstein space.*

Since a 3-dimensional Einstein space is of constant curvature, we have

Proposition 7. *If a complete, simply connected K-contact Riemannian manifold M is 3-dimensional and the Ricci tensor field is parallel, then M is globally isometric with a unit sphere.*

In [5] we have proved the following

Lemma 8. *Every conformally flat K-contact Riemannian manifold* (dim $M>3$) *is of constant curvature.*

In § 5, we prove

Lemma 9. *If* dim $M=3$ *and M is a conformally flat K-contact Riemannian manifold, then it is of constant curvature.*

Then we get

Theorem 10. *If M is a complete, simply connected and conformally flat K-contact Riemannian manifold, then M is globally isometric with a unit sphere.*

Theorem 1, Corollary 3, Corollary 6, and Lemma 8 are generalizations of the results on Sasakian manifolds obtained by M. Okumura [3].

§ 3. Proof of Theorem 1. Differentiating (1.3) covariantly, we have

(3.1) $$\phi_{hr}R^r{}_{jks}\xi^s-\eta_r R^r{}_{jks}\phi^s_h=-\phi_{hj}\eta_k-\phi_{hk}\eta_j.$$

Transvecting (3.1) with ϕ^h_i we get

(3.2) $$R_{ijks}\xi^s+R_{ikjs}\xi^s=2g_{jk}\eta_i-g_{ji}\eta_k-g_{ik}\eta_j.$$

We operate ∇_p again, and get

(3.3) $$R_{ijks}\phi^s_p+R_{ikjs}\phi^s_p=2g_{jk}\phi_{ip}-g_{ij}\phi_{kp}-g_{ik}\phi_{jp}.$$

Transvecting (3.3) with ϕ^p_l and using (3.2) we get

(3.4) $$R_{ijkl}+R_{ikjl}=2g_{jk}g_{il}-g_{ij}g_{kl}-g_{ik}g_{jl}.$$

Now we take an arbitrary point x of M and arbitrary orthonormal tangent vectors X and Y at x. If we contract (3.4) with $X^iY^jX^kY^l$, we have

(3.5) $$-R_{ijkl}X^iY^jX^kY^l=1.$$

This means that the sectional curvature is constant and equal to 1.

§ 4. Proof of Proposition 5. Suppose that the restricted homogeneous holonomy group ψ is reducible, then at any point x we have the direct decomposition of the tangent space M_x as

(4.1) $$M_x = L_x + N_x$$

such that $\psi(x)L_x \subset L_x$ and $\psi(x)N_x \subset N_x$. Let $X \in L_x$ and $Y \in N_x$, then we have $R(X, \xi)Y=0$, and by (1.3) we have $g(X, Y)-\eta(X)\eta(Y)=0$. Since $g(X, Y)=0$, we get $\eta(X)\eta(Y)=0$. Assume that $\eta(X)\neq 0$, then for any $Y \in N_x$, $\eta(Y)=0$ holds and this implies $\xi \in L_x$. By (1.3) again, any $Y \in N_x$ satisfies $g(Y, Y)=\eta(Y)\eta(Y)=0$. That is $L_x=(0)$.

§ 5. **Proof of Lemma 9.** When dim $M=3$, M is conformally flat if and only if

(5.1) $$C_{jkl} = \nabla_l R_{jk} - \nabla_k R_{jl} - (1/4)(g_{jk}\nabla_l S - g_{jl}\nabla_k S) = 0,$$

where S is the scalar curvature. Since ξ is a Killing vector field, it leaves R_{jk} and S invariant:

(5.2) $$L_\xi R_{jk} = \nabla_l R_{jk}\xi^l + R_{lk}\nabla_j \xi^l + R_{jl}\nabla_k \xi^l = 0,$$

(5.3) $$L_\xi S = \xi^l \nabla_l S = 0.$$

If we eliminate $\nabla_l R_{jk}$ from (5.1) and (5.2), using (1.2) and (5.3), we have

(5.4) $$-(m-1)\phi_{jk} - R_{rk}\phi_j^r = (1/4)\eta_j \nabla_k S.$$

Now we transvect (5.4) with ϕ_l^j, then we get $R_{kl}=(m-1)g_{kl}$. So M is an Einstein space, and of constant curvature.

References

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