153. *On the Principle of Limiting Amplitude*

By Nobuhisa IWASAKI

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§1. **Introduction and Theorem.** We study the behavior for large time of solutions of wave equations with a harmonic forcing term in the three dimensional euclidian space. That is called the principle of limiting amplitude. This principle states that every solution $u(x_1 t)$ for the initial value problem,

$$(1.1)\qquad \left\{\frac{\partial^2}{\partial t^2}+b(x)\frac{\partial}{\partial t}-\Delta+c(x)\right\}u(x, t)=f(x)e^{i\omega t}$$

$$(1.2)\qquad u(x, t)\Big|_{t=0}=\frac{\partial}{\partial x}u(x, t)\Big|_{t=0}=0,$$

tends to the steady state solution, $e^{i\omega t}v(x, i\omega)$, uniformly on bounded sets at $t\to\infty$. There $v(x, i\omega)$ satisfies the elliptic equation.

$$(1.3)\qquad \{-\Delta+c(x)+i\omega b(x)-\omega^2\}V(x, i\omega)=f(x),$$

and the Sommerfeld radiation conditions at infinity. In the case when $b(x)\equiv 0$ and the real valued function $c(x)$ is once continuously differentiable and its support is compact, this principle has been proved by D. A. Ladyzenskaja [1]. Here the rate of approach to steady state is like $e^{-\varepsilon t}$, $\exists\varepsilon>\sigma$, as $t\to\infty$. When $b(x)$ and $c(x)$ satisfy that $b(x)\geq 0$, $b(x)=\frac{1}{|x|^{3+\varepsilon}}$, $c(x)=\frac{1}{|x|^{2+\varepsilon}}$ as $|x|\to\infty$, and others, S. Mizohata and K. Mochizuki [2] has shown the principle, but they did not give the rate of approach. In this paper, we shall obtain the rate under the assumption that the real-valued function $b(x)\geq 0$, $c(x)\geq 0$ are bounded and their supports are compact.

Theorem. *Let $f(x)$ $b(x)$, and $c(x)$ be funtions which satisfy the following conditions.*

i) *$f(x)$, $b(x)$, and $c(x)$ vanish outside a bounded set*

ii) $\sum_{|\alpha|\leq 2}|D^\alpha f| \in L^2(E^3)$

iii) *$b(x)\geq 0$, $c(x)\geq 0$, and they are bounded functions.*

And let $u(x, t)$ be a solution for initial value problem (1.1), (1.2).

Then there exists a steady sate $e^{-i\omega t}V(x)$, such that

$$(1.4)\qquad \max_{x\in k}|u(x, t)-V(x)e^{i\omega t}|\leq C\cdot e^{-\varepsilon t},\ \exists\varepsilon>0 \text{ as } t\to\infty,$$

and V is a solution (1.3) *satisfying the Sommerfeld radiation conditions at infinity. Here K is a bounded set of E^3.* We can regard a solution $u(x, t)$ as a twice continuously differentiable function $u(t)$ from $[0, \infty)$ to $L^2(E^3)$ and as a continuous function to $\mathcal{E}^2_{L^2}(E^3)$. In this sense there exists the unique solution of (1.1), (1.2) if

$f(x) \in \varepsilon^1_{L^2}(E^3)$.

Let $\tilde{u}(\lambda)$ be the Laplace image of $u(t)$ with respect to t

(1.5) (ie) $$\tilde{u}(\lambda)=\int_0^\infty e^{-\lambda t}u(t)dt \quad \text{in } L^2$$

then

$$\tilde{u}(\lambda)=v(\lambda)/\lambda-i\omega$$

and

(1.6) $$u(t)=\frac{1}{2i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\frac{v(\lambda)}{\lambda-i\omega}e^{\lambda t}d \quad \text{in } L^2$$

for large enough $\sigma>0$.

Where $\{-\Delta+c(x)+\lambda b(x)+\lambda^2\}v(\lambda)=f(x)$, $v(\lambda)\in L^2$. $\mathrm{Re}\,\lambda>0$. Therefore we study the analyticity of $v(\lambda)$ with respect to λ and the order $\|v(\lambda)\|_{L^2(K)}$ as $|\mathrm{Im}\,\lambda|\to\infty$.

§2. **Some Lemmas.** 1) In the case when $b(x)\equiv c(x)\equiv 0$.

$$\{-\Delta+\lambda^2\}v(x,\lambda)=f(x),\ f(x)\in L^2$$

has the unique solution $v(x,\lambda)$ in $\varepsilon^2_{L^2}$ at $\mathrm{Re}\,\lambda>0$ and $v(x,\lambda)$ is an analytic function of λ to L^2.

$V(x,\lambda)\equiv R(x)f$ is represented by a fundamental solution $E(\lambda)$ as following

$$R(\lambda)f=E(\lambda)^*f,\ \text{where } E(\lambda)=\bar{F}\left\{\frac{1}{4\pi^2|\xi|^2+\lambda^2}\right\}=\frac{e^{-\lambda|x|}}{4\pi|x|}.$$

Let $Q(\delta)$ denote a Hilbert space consisting of all fuctions f such that $e^{\delta|x|}f\in L^2(E)$ with the inner product $(f,g)_\delta=(e^{\delta|x|}f, e^{\delta|x|}g)_{L^2(E^3)}$, $(-\infty<\delta<+\infty)$. Now it is clear that $Q(\delta)\subset Q(\delta')$ if $\delta>\delta'$.

Using these spaces

Lemma 1. *Let,* $R(\lambda)f=\dfrac{1}{4\pi}\displaystyle\int_{E^3}\frac{e^{-\lambda|x-y|}}{|x-y|}f(y)dy.$

Then $R(\lambda)$, which values a bounded operator from $Q(2\delta)$ to $Q(-2\delta)$, is an analytic function of λ and satisfies the following estimates at $\mathrm{Re}\,\lambda\geq-\delta$ $(\delta>0)$.

i) $|R(\lambda)f|_{-2\delta}\leq C/(1+|\lambda|)(1+|\mathrm{Re}\,\lambda|)\cdot|f|_{2\delta}$

ii) $|DR(\lambda)f|_{-2\delta}\leq C/(1+|\mathrm{Re}\,\lambda|)\cdot|f|_{2\delta}$

iii) $|D^2R(\lambda)f|_{-2\delta}\leq C(1+|\lambda|)/(1+|\mathrm{Re}\,\lambda|)\cdot|f|_{2\delta}$

iv) $|\{R(\lambda)-R(\lambda+h)\}f|_{-2\delta}\leq C|h|/(1+|\lambda|)(1+|\mathrm{Re}\,\lambda|)\cdot|f|_{2\delta}$,

$0<h<1$ *where* $|\ |_\delta$ *denote the norm of* $Q(\delta)_{(ie)}|f|^2_\delta=\int_{E^3}|e^{\delta|x|}f|^2dx.$

2) In the case when $b(x)\equiv 0$, $c(x)\not\equiv 0$.

Lemma 2. *Let* $L_1(\lambda)u=\{-\Delta+\lambda^2+c(x)\}u$, $u\in\varepsilon^2_{L^2}$ *and $G_1(\lambda)$ be the green operators of $L_1(\lambda)$*

(ie) $$G_1(\lambda)\cdot L_1(\lambda)\subset L_1(\lambda)\cdot G_2(\lambda)=I;\ L^2\to L^2$$

then we can consider $G_1(\lambda)$ as bounded operators from $Q(\delta)$ to $Q(-\delta)$. In this sense we can analytically continue $G_1(\lambda)$ to analytic function of λ at $\mathrm{Re}\,\lambda\geq-\delta'<0$, *which satisfies the following estimates.*

i) $|G_1(\lambda)f|_{-\delta}\leq\dfrac{c}{(1+|\lambda|)(1+|\mathrm{Re}\,\lambda|)}|f|_\delta$

ii) $|\{G_1(\lambda)-G_1(\lambda+h)\}f|_{-\delta}\leq c|h|/(1+|\lambda|)(1+|\mathrm{Re}\,\lambda)|\cdot|f|\delta$

and $G_1(\lambda)$ *are compact operators from* $Q(\delta)$ *to* $Q(-\delta)$ (*which mapp any bounded set to a precompact set*) *where* $c(x)\geq 0$ *is a bounded function with compact support.*

3) In the case when $b(x)\not\equiv 0$

Lemma 3. *Let* $L_2(\lambda)u=\{-\Delta+\lambda^2+c(x)+\lambda b(x)\}u$, $u\in\varepsilon_{L^2}^2$ *and* $G_2(\lambda)$ *be the green operators of* $L_2(\lambda)$.

(ie) $$G_2(\lambda)\cdot L_2(\lambda)\subset L_2(\lambda)\cdot G_2(\lambda)=I\colon\ L^2\to L^2$$

then we can also consider $G_2(\lambda)$ *as bounded operators from* $Q(\delta)$ *to* $Q(-\delta)$. *In this sense we can continue* $G_2(\lambda)$ *to an analytic function of* λ *at* $\mathrm{Re}\,\lambda\geq-\delta''<0$, *which satisfies the following estimate*

$$G_2(\lambda)f|_{-\delta}\leq\frac{c}{(1+|\lambda|)(1+|\mathrm{Re}\,\lambda|)}|f|_\delta$$

where $b(x)\geq 0$ *and* $c(x)\geq 0$ *is bounded functions with compact supports.*

(Proof of Lemma 1).

Since D, (c^∞-functions with compact support) is a dense subset of $Q(\delta)$, we may assume that $f\in D$ in order to prove the eitimates. It is clear that $R(\lambda)$ is an analytic function of λ at $\mathrm{Re}\,\lambda\geq-\delta$, which values the vector space consisting bounded operators from $Q(2\delta)$ to $Q(-2\delta)$. We show only the case $\lambda=a+ib$, $|a|\leq\delta$, $b\geq N>0$.

At $\mathrm{Re}\,\lambda>0$,

$$R(\lambda)f=\bar{F}\left[\frac{1}{(2\pi|\xi|)^2+\lambda^2}F(f)\right]$$
$$=\sum_{(i,j,k)}\int_{\Gamma(i,j,k)}\frac{e^{2\pi i x\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi$$

where F and $\bar{F}$ are Fourier and Fourier inverse transform, respectively.

i, j, and k take a sign $+$ or $-$, and

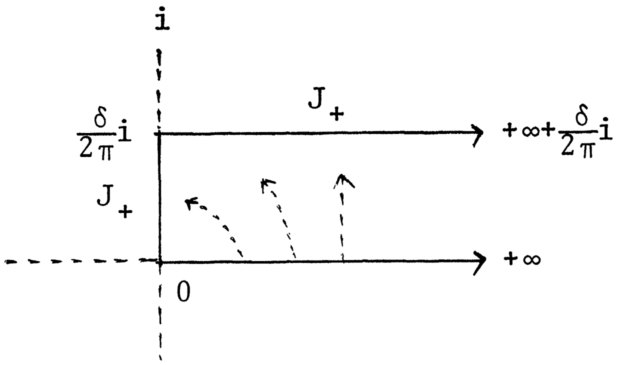
$$\Gamma(+++)=[0,\infty)\times[0,\infty)\times[0,\infty)$$
$$\Gamma(++-)=[0,\infty)\times[0,\infty)\times(-\infty,0]$$
$$\vdots$$

Let $R(ijk)(\lambda)f=\displaystyle\int_{\Gamma(ijk)}\frac{e^{2\pi i x\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi.$

Since we can consider that ξ is of three dimension complex number space: C^3, we may change the integral paths $\Gamma(ijk)$ as fllows,

$$[0,\infty)\to\left[0,+\frac{\delta}{2\pi}i\right]+\left[+\frac{\delta}{2\pi}i,\infty+\frac{\delta}{2\pi}i\right)=I_++J_+$$

$$(-\infty,0]\to\left[0,-\frac{\delta}{2\pi}i\right]+\left[-\frac{\delta}{2\pi}i,-\infty-\frac{\delta}{2\pi}i\right)=I_-+J_-$$



then

$$R(ijk)(\lambda)f \equiv \int_{(I_i+J_i)\times(I_j+J_j)\times(I_k+J_k)} \frac{e^{2\pi i x\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi.$$

The right side of the above equation is analytic at $\mathrm{Re}\,\lambda > -\delta$ and $\mathrm{Im}\ \lambda \geq N > 0$, therefore

$$R(\lambda)f = \sum_{(i,j,k)} \int_{(I_i+J_i)\times(I_j+J_j)\times(I_k+J_k)} \frac{e^{2\pi i x\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi$$

at $\mathrm{Re}\,\lambda > -\delta$, $\mathrm{Im}\,\lambda \geq N > 0$.

Thus we have only to prove that every term of the right side of the above satisfies the estimates of Lemma 1. In this place we estimate only a term which is

$$S(J_+, J_+, J_+)(\lambda)f \equiv \int_{J_+\times J_+\times J_+} \frac{e^{2\pi i x\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi$$

$$S(J_+, J_+, J_+)(\lambda)f = e^{-\delta(x_1+x_2+x_3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{2i(x_1s_1+x_2s_2+x_3s_3)}\frac{g(s_1, s_2, s_3)}{p(s_1, s_2, s_3\lambda)}ds_1ds_2ds_3$$

where

$$g(s_1s_2, s_3) = \hat{f}\left(s_1+i\frac{\delta}{2}, s_2+i\frac{\delta}{2}, s_3+i\frac{\delta}{2}\right).$$

$$p(s_1s_2s_3, \lambda) = 4\pi^2\left\{\left(s_1+i\frac{\delta}{2}\right)^2+\left(s_2+i\frac{\delta}{2}\right)^2+\left(s_3+i\frac{\delta}{2}\right)^2\right\}+\lambda^2.$$

Considering as Fourier transform from (s_1, s_2, s_3) to (x_1, x_2, x_3), we apply the Plancherel's theorem to $e^{\delta(x_1+x_2+x_3)}S(J_+J_+J_+)(\lambda)f$.

$$\int_{R^3}|e^{\delta(x_1+x_2+x_3)}S(J_+J_+J_+)(\lambda)f|^2dx$$
$$\leq \frac{c}{\inf|p(s_1s_2s_3\lambda)|^2}\int_{R^3}|g(s_1, s_2, s_3)|^2ds_1ds_2ds_3$$
$$0 \leq s_1, s_2, s_3 < \infty$$
$$\leq \frac{c'}{\inf|p(s_1s_2s_3\lambda)|^2}\int|e^{-\delta(x_1+x_2+x_3)}f(x)|^2dx$$
$$0 \leq s_1, s_2, s_3 < \infty$$

and

$$\inf_{\substack{0\le s_1,\, s_2,\, s_3<\infty}} | P(s_1, s_2, s_3, \lambda)\geq \frac{1}{2} \inf_{0\le t<\infty} | (t+i\delta)^2+\lambda^2 |$$
$$\geq c(1+|\lambda|)(\delta+\mathrm{Re}\,\lambda)$$

therefore

$$\int | e^{-(\sqrt{3}\delta+\varepsilon)|x|}S(J_+, J_+, J_+)(\lambda) f |^2 dx$$
$$\leq \frac{c}{(1+|\lambda|)^2(\delta+\mathrm{Re}\,\lambda)^2}\int_{R^3} | e^{(\sqrt{3}\delta+\varepsilon)|x|} f(x) |^2 dx. \qquad \text{q.e.d.}$$

In order to prove Lemma 2, it is sufficient to solve the equation of operations,

$$\{I+R(\lambda)\cdot c(x)\}G_1(\lambda)=R(\lambda)$$

that is, to show the existence of $\{I+R(\lambda)\cdot c(x)\}^{-1}$ on $Q(-\delta)$ $\mathrm{Re}\,\lambda\geq-\delta'<0$. This follows, when $|\mathrm{Im}\,\lambda|$ is sufficiently large, from the existence of inverse by Newmann series using Lemma 1. i), and, when $|\mathrm{Im}\,\lambda|$ is finite, from the fact that the self-adjoint operator $L_1(0)$ has no discreat eigen value and $R(\lambda)\cdot c(x)$ is a compact operator on $Q(-\delta)$.

In order to prove Lemma 3 we also solve the equation

$$\{I+\lambda G_1(\lambda)\cdot b(x)\cdot\}G_2(\lambda)=G_1(\lambda)$$

that is equivalent to showing the existence of inverse of $\{I+T_\lambda\}$ on L^2, where

$$T_\lambda=\lambda\cdot a(x)\cdot G_1(\lambda)\cdot a(x),\ \ a(x)^2\equiv b(x),$$

and $G_2(\lambda)$ is given by $\{-\lambda G_1(\lambda)a(x)(I+T_\lambda)^{-1}+I\}G_1(\lambda)$. By the same method of S. Mizohata, K. Mochizuki [2],

$$\| v \|_{L^2}<\|\{I+T_\lambda\}v\|_{L^2} \quad \text{at} \quad \mathrm{Re}\,\lambda\geq 0$$

because

$$\mathrm{Re}\,(T_\lambda v, v)=\int_\delta^\infty \frac{(\mu+a^2+b^2)a}{(\mu+a^2-b^2)^2+(2ab)^2}\,d(E_\mu a(x)v, a(x)v)\geq 0 \quad \text{q.e.d.}$$

where E_μ is the resolution of the identity of the positive self-adjoint operator $L_1(0)$, and $\| v \|_{L^2}^2+\mathrm{Re}\,(T_\lambda v, v)=\mathrm{Re}\,(\{I+T_\lambda\}v, v)$. Therefore $G_2(\lambda)$ exists at $\mathrm{Re}\,\lambda\geq 0$ and satisfies the estimate of Lemma 3. Since $G_1(\lambda)$ satisfies the estimate of Lemma 2, ii) we can extend the domain of existence of $G_2(\lambda)$ to $\mathrm{Re}\,\lambda\geq-\delta''<0$ by Neumann series.

Proof of Theorem from (1.6)

$$u(t)=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\frac{v(\lambda)}{\lambda-i\omega}d\lambda \quad \text{in} \quad L^2, \sigma>0, v(\lambda)=G_2(\lambda)f.$$

By Lemma 3, we can use the Cauchy integral formula.

We obtain that

$$u(t)=\frac{1}{2\pi i}\int_{-\varepsilon\pm i\infty}^{-\varepsilon+i\infty}\frac{v(\lambda)}{\lambda-i\omega}d\lambda+G_2(i\omega)fe^{i\omega t},\ \exists\varepsilon>0.$$

Considering that $\sum_{|\alpha|\leq 2}|D^\alpha f|\in L^2(E^3)$ we have the estimate that

$$\max_{x \in k} |v(\lambda)| \leq \frac{c}{1+|\lambda|}, \quad \mathrm{Re}\,\lambda \geq -\varepsilon$$

Thus we conclude (1.4).

References

[1] O. A. Ladyzenskaja: On the principle of limiting amplitude. Uspehi Math. Nauk, **12** (3), 161-164 (1957).

[2] S. Mizohata and K. Mochizuki: On the principle of limiting amplitude for dissipative wave equations. Jour. Math. Kyoto Univ., **6** (1) (1966).