190. *Partially Ordered Sets and Semi-Simplicial Complexes*

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§ 1. **Introduction.** Let $\mathcal{M}$ be the category of partially ordered sets and isotone maps, and $\mathcal{S}$ the one of s.s. (semi-simplicial) complexes and s.s. maps.

Then, a covariant functor L: $\mathcal{M}\to\mathcal{S}$ is defined naturally as follows:

For a partially ordered set X, let $M(X)$ be the ordered simplicial complex whose n-simplex is an ordered sequence $(x_0, x_1, \cdots, x_n)$ for $x_i \in X$ and $x_0 < x_1 < \cdots < x_n$, and define $L(X)$ as the ordered s.s. complex of $M(X)$.

The object of this note is to discuss on the fundamental properties of L. It is shown that two partially ordered sets X and Y are isomorphic if and only if $L(X)$ and $L(Y)$ are s.s. isomorphic (Corollary 6). Also, we can define the notion of "homotopy" so that X and Y are homotopy equivalent if and only if $L(X)$ and $L(Y)$ are so (Corollary 8).

Furthermore, a (co)homology group of a pair (X, A) of a partially ordered set and its ideal can be defined by the one of the s.s. pair $(L(X), L(A))$, and the seven axioms of Eilenberg-Steenrod ([2]) are satisfied (Theorem 10).

It is interesting that $L(X)$ satisfies the extension condition for the dimension >1(Theorem 4). Here, we notice that there does not necessarily exist a partially ordered set X such that $M(X)$ is simplicially isomorphic to a given simplicial complex K.

Full details will be appear elsewhere.

§ 2. **Fundamental properties of L.** For the terminology and the notations concerning the partially ordered sets or the s.s. (semi-simplicial) complexes, see [1] or [5] respectively.

For a partially ordered set X, and s.s. complex $L(X)$ is defined as follows:

An n-simplex of $L(X)$ is an ordered sequence $(x_0, \cdots, x_n)$ where $x_i \in X$ and $x_0 \leqq \cdots \leqq x_n$. The face- and degeneracy-operators are given by

$$\partial_i(x_0, \cdots, x_n) = (x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$
$$s_i(x_0, \cdots, x_n) = (x_0, \cdots, x_i, x_i, \cdots, x_n),$$

where $i=0, 1, \cdots, n$.

Clearly, $L(X)$ is the ordered s.s. complex of the ordered simplicial complex $M(X)$ defined in § 1, and so the geometric realization ([4]) of $L(X)$ is the polyhedron $|M(X)|$.

For an isotone map $f: X \to Y$ between partially ordered sets X and Y, an s.s. map $L(f): L(X) \to L(Y)$ is defined by

$$L(f)(x_0, \cdots, x_n) = (f(x_0), \cdots, f(x_n)),$$

and its realization $|L(f)|$ is clearly the simplicial map of $|M(X)|$ to $|M(Y)|$ which carries any vertex (x) to $(f(x))$. Immediately we have

Lemma 1. *Let A be an ideal of X and $i: A \to X$ be the inclusion map. Then $L(A)$ is a subcomplex of $L(X)$, and $L(i)$ is the inclusion map.*

Lemma 2. *$L(gf)=L(g)L(f)$, for isotone maps $f: X \to Y$ and $g: Y \to Z$.*

By these lemmas, we have

Theorem 3. *L is the covariant functor of $\mathcal{M}$ to $\mathcal{S}$.*

Theorem 4. *For a partially ordered set X and any integer $m>1$, $L(X)$ satisfies the following condition.*

$K(m)$: Let $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_{m+1}$ be m-simplices of $L(X)$ such that $\partial_i \sigma_j = \partial_{j-1} \sigma_i$ for $i<j$, then there exists a unique $(m+1)$-simplex of $L(X)$ such that $\partial_i \sigma = \sigma_i$ for $i \neq k$.

This condition $K(m)$ is known as the extension condition ([3]), except the uniqueness of σ.

A pair (X, A) of a partially ordered set X and its ideal A will be called simply a *pair*. For pairs (X, A) and (Y, B), the set of all isotone maps of (X, A) to (Y, B) is denoted by Ist $((X, A),(Y, B))$. Also, for s.s. pairs (K, N) and (K', N'), the set of all s.s. maps of (K, N) to (K', N') is denoted by Map $((K, N), (K', N'))$.

Theorem 5. *For any pairs (X, A) and (Y, B), the correspondence L:* Ist $((X, A), (Y, B)) \to$ Map $((L(X), L(A)), (L(Y), L(B)))$, *given by $f \to L(f)$, is one-to-one and onto.*

Proof. Clearly L is one-to-one. We show that L is onto. Let $\varphi: (L(X), L(A)) \to (L(Y), L(B))$ be a s.s. map. For any $x \in X$, define $f(x) \in Y$ by $(f(x))=\varphi(x)$. This f is an isotone map of (X, A) to (Y, B) and $L(f)=\varphi$.

Corollary 6. *If $L(X)$ is s.s. isomorphic to $L(Y)$, then X is isomorphic to Y.*

§ 3. Ordered homotopy. Let (X, A) and (Y, B) be two pairs. For $f, g \in$ Ist $((X, A), (Y, B))$, f is said to be *ordered homotopic* to g, if there exists a finite number of maps $h_1, \cdots, h_n \in$ Ist $((X, A), (Y, B))$ such that $f=h_1$, $g=h_n$, and $h_{i-1}=h_i$ or $h_i=h_{i-1}$ for $i=2, \cdots, n$. This relation is clearly an equivalence relation in Ist $((X, A), (Y, B))$.

Let (K, N) and (K', N') be two s.s. pairs. For $\varphi, \psi \in \mathrm{Map}((K, N), (K', N'))$, φ is said to be *semi-homotopic* to ψ if there exists a finite number of maps $\varphi_1, \cdots, \varphi_n \in \mathrm{Map}((K, N), (K', N'))$ such that $\varphi = \varphi_1$, $\psi = \varphi_n$ and φ_{i-1} or φ_i is s.s. homotopic to φ_i or φ_{i-1} respectively. This is an equivalence relation in $\mathrm{Map}((K, N), (K', N'))$.

Theorem 7. *For partially ordered sets X and Y, let f and g be two isotone maps of X to Y. Then, f is ordered homotopic to g if and only if $L(f)$ is semi-homotopic to $L(g)$.*

Proof. Assume that $f \leqq g$. Then a s.s. homotopy $F: L(X) \times \triangle_1 \to L(Y)$ is defined by

$$F((x_0, \cdots, x_n) \times (\underbrace{0, \cdots, 0}_{i}, 1, \cdots, 1)) = (f(x_0), \cdots, f(x_i), g(x_{i+1}), \cdots, g(x_n))$$

for any n-simplex $(x_0, \cdots, x_n)$ of $L(X)$. Thus $L(f)$ is s.s. homotopic to $L(g)$.

Conversely, assume that $L(f)$ is s.s. homotopic to $L(g)$, and $F: L(X) \times \triangle_1 \to L(Y)$ be its homotopy. Then, for any $x \in X$, we have $F((x) \times (0)) = (f(x))$ and $F((x) \times (1)) = (g(x))$, and so

$$F((x, x) \times (0, 1)) = (f(x), g(x)).$$

This shows that $f \leqq g$.

Corollary 8. *The correspondence L of Theorem 5 induces the one-to-one correspondence of the set of all ordered homotopy classes of* $\mathrm{Ist}((X, A), (Y, B))$ *onto the set of all semi-homotopy classes of* $\mathrm{Map}((L(X), L(A)), (L(Y), L(B)))$.

Corollary 9. *If two isotone maps f and g of (X, A) to (Y, B) are ordered homotopic, then the continuous maps $|L(f)|$ and $|L(g)|$ of $(|L(X)|, |L(A)|)$ to $(|L(Y)|, |L(B)|)$ are homotopic.*

Now define a (co)homology group of a pair (X, A) to be the one of the s.s pair $(L(X), L(A))$.

Theorem 10. *This (co)homology theory on the category of all pairs of partially ordered sets satisfies the seven axioms of Eilenberg-Steenrod ([2]), where the notion of "homotopic" is the one of "ordered homotopic".*

§ 4. Remarks. Related to Corollary 6, we notice that there exist two partially ordered sets X and Y which are not isomorphic but the simplicial complex $M(X)$ and $M(Y)$ are simplicially isomorphic. Let $X = \{a_0, a_1, a_2, a_3\}$ where $a_0 < a_1 < a_3$ and $a_0 < a_2 < a_3$, and $Y = \{b_0, b_1, b_2, b_3\}$ where $b_0 < b_1 < b_2$ and $b_1 < b_3$. Then $M(X)$ is simplicially isomorphic to $M(Y)$.

Let $Z(n)$ be the linearly ordered set with $n+1$ elements $0 < 1 < \cdots < n$. Then $L(Z(n))$ can be considered naturally as the standard simplex $\triangle_n$. It is important that the standard simplex for any order type is generally defined by this way.

Also, we notice that, for a given simplicial complex K, there does not necessarily exist a partially ordered set X such that $M(X)$ is simplicially isomorphic to K. For example, if K is the boundary complex $\dot{\triangle}_n$ of the standard simplex $\triangle_n$, then K has no such X. However we have

Theorem 11. *For any simplicial complex K, there exists a partially ordered set X such that $|M(X)|$ is homeomorphic to $|K|$.*

References

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