1. *On Certain Square Integrable Irreducible Unitary Representations of Some $\mathfrak{P}$-Adic Linear Groups*

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(Comm. by Zyoiti SUETUNA, M.J.A., Jan. 12, 1968)

0. **Introduction.** Let P be a $\mathfrak{P}$-adic number field. Denote by $\mathcal{O}$, $\mathfrak{P}$, and $\mathcal{O}^*$ the ring of integers, the maximal ideal of $\mathcal{O}$ and the unit group respectively. Mautner proved that the $PGL(2, P)$ has square integrable irreducible unitary representations induced by certain irreducible representations of some maximal compact subgroup of $PGL(2, P)$.

In this note, we shall consider the subgroup G of $GL(n, P)$ formed by the matrices with determinant in $\mathcal{O}^*$. Using the theory of induced representations of finite groups, we first construct irreducible unitary representations of $K = GL(n, \mathcal{O})$ parametrized by certain characters of the unit group of the unramified extension of P of degree n, which are monomial if n is odd. Modifying the method of Mautner, we shall show that the representations of G induced by above representations of K are square integrable and irreducible. For simplicity we assume that n is odd. But we can construct similar representation when n is even, though the result becomes somewhat complicated. Modifying Harish-Chandra's character formula for square integrable representations of real semi-simple Lie groups, we can get a character formula for our representations. Similar results can be obtained for $SL(n, P)$.

The author could get copies of J.A. Shalika's lectures in seminar on representations of Lie groups held at Princeton in 1966.* The author's work is independent of Shalika's results. But their method and results overlap each other to a certain extent. Detailed proofs will be published elsewhere.

1. For any integer n we denote by $P^{(n)}$ the unramified extension of P of degree n. Let $\mathcal{O}^{(n)}$ be the ring of integral elements of $P^{(n)}$ and $\mathcal{P}^{(n)}$ be the maximal ideal of $\mathcal{O}^{(n)}$. Let π be a generator of $\mathcal{P}$ in $\mathcal{O}$. Then π is a generator of $\mathcal{P}^{(n)}$ in $\mathcal{O}^{(n)}$. We denote by $\mathrm{Gal}(P_n/P)$ the Galois group of $P^{(n)}$ with respect to P. $\mathrm{Gal}(P_n/P)$ is a cyclic group of order n. Let σ be a generator of this group. Let J be the following matrix in $M(n, \mathcal{O}^{(n)})$:

*) J. A. Shalika: Representations of the two by two unimodular groups over local fields. I, II.

$$J=\begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & 0 & & & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

We introduce following subgroups of $GL(n, P^{(n)})$:

$$\mathcal{G}=\{g \in GL(n, P^{(n)});\ g^{\sigma}=JgJ^{-1},\ \det g \in \mathcal{O}^*\},$$

$$\mathcal{K}=\mathcal{G} \cap M(n, \mathcal{O}^{(n)}),$$

$$\mathcal{A}=\left\{\begin{pmatrix} \alpha & & & \\ & \sigma\alpha & & \\ & & \ddots & \\ & & & \sigma^{n-1}\alpha \end{pmatrix};\quad \alpha \in \mathcal{O}^{(n)*}\right\},$$

(Here we denote by $\mathcal{O}^{(n)*}$ the unit group of $\mathcal{O}^{(n)}$.)

$$\mathcal{K}_l=\{k \in \mathcal{K};\ k-1 \in \pi^l M(n, \mathcal{O}^{(n)})\},$$

$$\mathcal{J}_l=\mathcal{A}\cdot\mathcal{K}_l \qquad (l=1, 2, \cdots).$$

For any $z_1, z_2, \cdots, z_n \in P^{(n)}$ we denote by $\mathfrak{M}(z_1, z_2, \cdots, z_n)$ the following matrix in $M(n, P^{(n)})$:

$$\mathfrak{M}(z_1, z_2, \cdots, z_n)=\begin{pmatrix} z_1 & z_2 & \cdots & \cdots & z_n \\ \sigma z_n & \sigma z_1 & \sigma z_2 & & \\ & \sigma^2 z_n & \ddots & \ddots & \\ & & \ddots & \ddots & \sigma^{n-2}z_2 \\ \sigma^{n-1}z_2 & & & \sigma^{n-1}z_n & \sigma^{n-1}z_1 \end{pmatrix}.$$

We have

$$\mathcal{K}_l=\{\mathfrak{M}(1+\pi^l z_1, \pi^l z_2, \cdots, \pi^l z_n);\ z_1, z_2, \cdots, z_n \in \mathcal{O}^{(n)}\}.$$

It is easily seen that $\mathcal{G}$ (resp. $\mathcal{K}$) is isomorphic to G (resp. K). In the following we assume for simplicity that n is odd ($n=3, 5, 7, \cdots$). We define the subgroup $\mathcal{H}_l$ of $\mathcal{K}_l$ as follows:

$$\mathcal{H}_l=\{\mathfrak{M}(1+\pi^l z_1, \pi^l z_2, \cdots, \pi^l z_{\frac{n+1}{2}}, \pi^{l+1} z_{\frac{n+1}{2}+1}, \cdots, \pi^{l+1} z_n);\ z_1, \cdots, z_n \in \mathcal{O}^{(n)}\}.$$

We put $\mathcal{J}_l=\mathcal{A}\mathcal{H}_l$. Then $\mathcal{J}_l$ is a subgroup of $\mathcal{J}_l$. $\mathcal{A}$ is a compact abelian group isomorphic with the unit group of $\mathcal{O}^{(n)}$. Let ξ be a non trivial character of $\mathcal{A}$. There exists a natural number $f=f(\xi)$ such that ξ is identically equal to 1 on $\mathcal{A} \cap \mathcal{K}_f$ but not identically equal to 1 on $\mathcal{A} \cap \mathcal{K}_{f-1}$. We call f conductor of ξ. $\mathrm{Gal}(P^{(n)}/P)$ operates naturally on the character group of $\mathcal{A}$ as follows:

$$\xi^{\tau}(a)=\xi(\tau a) \qquad (\tau \in \mathrm{Gal}(P^{(n)}/P)).$$

We call ξ regular if $f(\xi(\xi^{\tau})^{-1})=f(\xi)$ for any $1 \neq \tau \in \mathrm{Gal}(P^{(n)}/P)$.

Lemma 1. *When $f=f(\xi)$ is even, we define a function μ_{ξ} on $\mathcal{J}_{\frac{f}{2}}$ as follows* $\left(\textit{we put } l=\dfrac{f}{2}\right)$:

$$\mu_{\xi}(a\mathfrak{M}(1+\pi^l z_1, \pi^l z_2, \cdots, \pi^l z_n))=\xi(a)\xi(1+\pi^l z_1)$$

$(a \in \mathcal{A}, z_1, \cdots, z_n \in \mathcal{O}^{(n)})$.

Then μ_ξ is a 1-dimensional character of $\mathcal{G}_{\frac{f}{2}}$.

Lemma 2. *When $f=f(\xi)$ is odd and $f \geqslant 3$, we define a function ν_ξ on $\mathcal{G}_{\frac{f-1}{2}}$ as follows* $\left(\text{we put } l=\frac{f-1}{2}\right)$:

$$\nu_\xi(a\mathfrak{M}(1+\pi^l z_1, \pi^l z_2, \cdots, \pi^l z_{\frac{n+1}{2}}, \pi^{l+1} z_{\frac{n+1}{2}+1}, \cdots, \pi^{l+1} z_n))$$
$$=\xi(a)\xi(1+\pi^l z_1) \qquad (a \in \mathcal{A}, z_1, \cdots, z_n \in \mathcal{O}^{(n)}).$$

Then ν_ξ is a 1-dimensional character of $\mathcal{G}_{\frac{f-1}{2}}$.

Let ξ be a regular character of $\mathcal{A}$. We assume $f=f(\xi) \geqslant 2$. We define a representation U_ξ of $\mathcal{K}$ as follows:

$$U_\xi = \operatorname*{Ind}_{\mathcal{G}_{\frac{f}{2}} \uparrow \mathcal{K}} \mu_\xi \qquad \text{when } f \text{ is even},$$

$$U_\xi = \operatorname*{Ind}_{\mathcal{G}_{\frac{f-1}{2}} \uparrow \mathcal{K}} \nu_\xi \qquad \text{when } f \text{ is odd}.$$

Theorem 1. *U_ξ is an irreducible unitary representation of $\mathcal{K}$. The dimension of the representation space is*

$$q^{\frac{n(n-1)}{2}(f-1)}(q-1)(q^2-1)\cdots(q^{n-1}-1).^{*)}$$

U_ξ and $U_{\xi'}$ are mutually equivalent if and only if there exists $\tau \in \mathrm{Gal}\,(P_n/P)$ such that $\xi'=\xi^\tau$.

Theorem 2. *Let T_ξ be the representation of $\mathcal{G}$ induced by the representation U_ξ of $\mathcal{K}$. Then T_ξ is a square integrable irreducible unitary representation of $\mathcal{G}$. T_ξ and $T_{\xi'}$ are mutually equivalent if and only if U_ξ and $U_{\xi'}$ are mutually equivalent.*

Let dg be the Haar measure on $\mathcal{G}$ such that $\int_{\mathcal{K}} dg=1$.

Theorem 3. *The formal degree of T_ξ is*

$$q^{\frac{n(n-1)}{2}(f-1)}(q-1)(q^2-1)\cdots(q^{n-1}-1).$$

The character of T_ξ is given as follows:

$$\mathrm{Trace}\ T_\xi(\varphi)=\int_{\mathcal{G}}\left(\int_{\mathcal{K}}\varphi(gkg^{-1})\ \mathrm{Trace}\ U_\xi(k)dk\right)dg$$

for any Schwartz-Bruhat function φ on $\mathcal{G}$.

The above character formula is an analogue of Frobenius' formula for induced characters.

References

[1] Harish-Chandra: Representations of semi-simple Lie groups. VI. Amer. J. Math., **78**, 564-628 (1956).

[2] Mautner, F.I.: Spherical functions over $\mathfrak{P}$-adic fields. II. Amer. J. Math., **86**, 171-200 (1964).

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*) We denote by q the order of $\mathcal{O}/\mathfrak{P}$.