58. Compactness and Completeness in Ranked Spaces

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The purpose of this note is to study the relation between the compactness and the completeness in ranked spaces.

The notion of the completeness in the ranked spaces was introduced in the note [1], and that of the compactness in the notes [3] and [6]. Every complete metric space can be considered a complete ranked space.

In any metric space, every sequentially compact set is complete. This assertion does not always hold in ranked spaces. In the first half of this note, we shall show it and give some condition which makes the assertion hold. And in the latter, we shall give some conditions with which the completeness yields the compactness. Throughout this note, we shall always treat ranked spaces with indicator ω_0 ([2], p. 319), and $m, n, \cdots$ will denote non-negative integers.

§ 1. From the compactness. A sequence $\{U_n(x_n)\}_{n=0,1,2,\dots}$ of subsets of a ranked space is called a *fundamental sequence* if it possesses the following three properties ([7] p. 1142):

(1) any $U_n(x_n)$ is a neighbourhood of point x_n of rank r_n, and

$$r_0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq \cdots, \text{ and } \lim_{n\to\infty} r_n = +\infty ;$$

(2) $U_0(x_0) \supseteq U_1(x_1) \supseteq U_2(x_2) \supseteq \cdots \supseteq U_n(x_n) \supseteq \cdots$;

(3) for any n, there is an m such that $r_m < r_{m+1}$, and that a neighbourhood $V(x_m)$ of point x_m of rank s $(r_m < s \leq r_{m+1})$ containing $U_{m+1}(x_{m+1})$ and included in $U_m(x_m)$ exists:

$$U_m(x_m) \supseteq V(x_m) \supseteq U_{m+1}(x_{m+1}).$$

A ranked space is said to be *complete* if, for every fundamental sequence $\{U_n(x_n)\}$, we have $\bigcap_n U_n(x_n) \neq \phi$.

Example 1. Let E be the interval $[0, 1]$ of real numbers and, for any x of E and for any n, let

$$\mathfrak{B}_{2n}(x) = \mathfrak{B}_{2n+1}(x) = \left\{ \left(x - \frac{1}{n},\ x + \frac{1}{n} \right) \cap E \right\}.^{1)}$$

Then E is a ranked space, and is r-compact ([6]). But it is not complete, because for the fundamental sequence $\{U_n(x_n)\}_{n=0,1,2,\dots}$ where

1) $\mathfrak{B}_n(x)$ will denote the family of neighbourhoods of point x and of rank n. [8] p. 616.

$x_n=\dfrac{1}{n+1}$ and $U_n(x_n)=\left(0, \dfrac{1}{n+1}\right)\in\mathfrak{B}_{2n+2}(x)$, we have $\bigcap_n U_n(x_n)=\phi$.

In this example, if we put $\mathfrak{B}_n(x)=\left\{\left(x-\dfrac{1}{n}, x+\dfrac{1}{n}\right)\cap E\right\}$, then E is complete.

Now let us introduce a new separation axiom:

(R_3) *for any two neighbourhoods $U(x)$ and $V(x)$ of any point x, if $U(x)\in\mathfrak{B}_m(x)$, $V(x)\in\mathfrak{B}_n(x)$ $(m<n)$ and if $U(x)\supseteq V(x)$, then we have $U(x)\supseteq \mathrm{cl}(V(x))$* ([6], p. 69).

Every example in the note [5] satisfies this axiom. It is not easy to show that Example 3 in it satisfies (R_3).

In this example, let $m<n$ and $u(m, p; 0)\supseteq v(n, q; 0)$. From the latter, we have $p\geq q$ ([5] p. 588). If $f_k\to f$ $(k\to\infty)$ in Φ', then, for any φ in Φ, $f_k(\varphi)\to f(\varphi)$ $(k\to\infty)$. Therefore, if all f_k's further belong to $v(n, q; 0)$, then we have $\|f\|_q'\leq\dfrac{1}{n}$. Hence, from $p\geq q$ and $m<n$, we have

$$\|f\|_p'\leq\|f\|_q'\leq\frac{1}{n}<\frac{1}{m},\quad \text{so}\quad f\in u(m, p; 0).$$

This proves that $u(m, p; 0)\supseteq \mathrm{cl}(v(n, q; 0))$.[2]

Theorem 1. *If a ranked space satisfying the axiom (R_3) is r-compact, then it is complete.*

Proof. Let $\{U_n(x_n)\}_{n=0,1,2,\dots}$ be a fundamental sequence. From the r-compactness, a subsequence $\{x_{n_k}\}_{k=0,1,2,\dots}$ of $\{x_n\}$ r-converges to a point x, so we have, for any m, $x\in \mathrm{cl}(U_m(x_m))$.

From the above property of the definition of fundamental sequence, and from the axiom (R_3), for every $U_n(x_n)$ there is an m such that $U_n(x_n)\supseteq \mathrm{cl}(U_m(x_m))$. Therefore $x\in\bigcap_n U_n(x_n)$. Hence, this ranked space is complete.

Let A be a subset of a ranked space E. We shall say that A is *complete*, if, for any fundamental sequence $\{U_n(x_n)\}$ in E where $x_n\in A$ $(n=0, 1, 2, \cdots)$, we have $\bigcap_n (U_n(x_n)\cap A)\neq\phi$.

If a subset A of a complete ranked space is closed under the paraconvergence ([3] p. 23 and [7] p. 1142) then A is complete.

Corollary. *In a ranked space satisfying the axiom (R_3), any r-compact subset is complete.*

§2. From the completeness. First, we shall study some properties of a ranked space E, in which every point x has essentially only one f.s. $\tau(x)$ to x[3] satisfying the following three conditions:

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2) We owe this proof to Miss Masako Washihara.

3) We shall call a fundamental sequence of neighbourhoods with respect to x ([2] p. 551) briefly an f. s. to x.

(4) for any n and for any point x of E, there is in $\tau(x)$ one and only one term of rank n, say $W_n(x)$;
(5) for any m, there is an n such that $y \in W_n(x)$ implies that $W_n(x) \subseteq W_m(y)$;
(6) if $x \in W_n(y)$, then $y \in W_n(x)$.

In short, E is also a symmetric uniform space, and $\tau(x)=\{W_n(x)\}$ is a base of the system of neighbourhoods of point x. For example, any countably normed space possesses this property ([5]).

Lemma 1. *This ranked space E is r-compact, if and only if the uniform space E is sequentially compact.*

As there is no loss of generality, we assume that n in the condition (5) is equal to $m+1$.

Lemma 2. *$y \in W_{n+1}(x)$ implies $W_{n+1}(y) \subseteq W_n(x)$, and $W_{n+2}(y) \cap W_{n+1}(x) \neq \phi$ implies $W_{n+2}(y) \subseteq W_n(x)$.*

Proposition 1. *E is a complete ranked space, if and only if it is a complete uniform space.*

Proof. Suppose that E is a complete ranked space. Let $\mathfrak{f}$ be a Cauchy filter on E, and, for any n, F_n be a member of $\mathfrak{f}$ which is covered by $W_{2n}(x)$ of its any point x: i.e. $F_n \subseteq W_{2n}(x)$. We may assume that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots .$$

For any n, let x_n be a point of F_{n+1}, and $U_n(x_n)=W_{2n}(x_n)$. Because $x_n \in F_{n+1} \subseteq F_n$, $F_n \subseteq U_n(x_n)$. Hence, from the above Condition (5) and from $x_{n-1} \in F_n$, we have

$U_n(x_n)=W_{2n}(x_n) \subseteq W_{2n-1}(x_{n-1}) \subseteq W_{2n-2}(x_{n-1})=U_{n-1}(x_{n-1})$, so $\{U_n(x_n)\}$ is a fundamental sequence of neighbourhoods. From the completeness of the ranked space E, there is a point y in $\bigcap_n U_n(x_n)$. We shall show that the filter $\mathfrak{f}$ converges to this y in the uniform space E.

For any positive n, $y \in U_n(x_n)=W_{2n}(x_n)$ implies that $U_n(x_n) \subseteq W_n(y)$, so $F_n \subseteq W_n(y)$. Hence $\mathfrak{f}$ converges to y in the uniform space E, so E is a complete uniform space.

Conversely, let E be a complete uniform space, and $\{U_n(x_n)\}$ be a fundamental sequence of neighbourhoods where $U_n(x_n)=W_{k_n}(x_n)$ ($n=0,1,2,\cdots$). We may suppose that $k_n < k_{n+1}$ and $W_{k_{n+1}}(x_n) \supseteq W_{k_{n+1}}(x_{n+1})$. From the above Condition (5), $\{U_n(x_n)\}$ is a base of a Cauchy filter and from the completeness of the uniform space E, there is a point y in E such that, for any n, $W_{k_{n+2}}(y)$ includes some $U_m(x_m)$ ($m>n$). As $U_m(x_m)=W_{k_m}(x_m) \subseteq W_{k_{n+1}}(x_n)$ and $k_m \geq k_n+1$ we have, from Lemma 2,

$$W_{k_{n+2}}(y) \subseteq W_{k_n}(x_n)=U_n(x_n).$$

Therefore $\bigcap_n U_n(x_n) \neq \phi$, so E is a complete ranked space.

We shall say that the ranked space E is *totally bouuded* if, for any n, it is covered by finite number of $W_n(x_1)$, $W_n(x_2)$, $\cdots$, $W_n(x_m)$.

The ranked space E is totally bounded if and only if E is a totally bounded uniform space.

Thus we have the following:

Theorem 2. *Suppose that a ranked space E, in which each point x has essentially only one f.s. $\tau(x)$ to x, satisfies the above Conditions* (4)$\sim$(6). *If E is complete and totally bounded, then it is r-compact.*

Corollary. *In a ranked space satisfying the supposition of Theorem* 2, *any complete and totally bounded subset is r-compact.*

Next, let E be a ranked space with the set T of indices, treated in the note [6], §2. For any index τ in T, the ranked space, consisting of the set E and of only $\tau(x)$'s ($x \in E$), will be denoted by E_τ, and the former will be also denoted by E_T.

Applying Theorem 2 to this ranked space E_T, we have the following:

Theorem 3. *Suppose that, for an index τ in T, E_τ satisfies the above Conditions* (4)$\sim$(6). *Then if E_τ is complete and totally bounded, then E_T is r-compact. And if a subset A of E is complete and totally bounded in E_τ, then A is r-compact in E_T.*

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