102. *Integration with Respect to the Generalized Measure. III*

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1. Introduction. Suppose M, $\mathcal{S}$, G, K, J, $\mathcal{F}$, $\mathcal{G}$, μ, and $\mathcal{J}$ are defined as are in the example in the introduction in [1]. Then $(\mathcal{S}, \mathcal{G}, J)$ is an abstract integral structure [1] and $\mathcal{J}$ is an abstract integral [1] with respect to this structure. For each $a \in K$, let $\bar{a}$ be the function in $\mathcal{F}$ such that $\bar{a}(x)=a$ for each $x \in M$. Then the operator "$-$" may be considered as an isomorphism of the topological additive group K into $\mathcal{F}$. Let us denote by $\bar{K}$ the image of K by this isomorphism. The topological additive group K can be identified with the subgroup $\bar{K}$ of $\mathcal{F}$ by this isomorphism and it holds that $K \subset \mathcal{G}$.

Now let i be the map of $\mathcal{S} \times K$ into J such that $i(X, \bar{a}) = \mu(X) \cdot a$ for each $X \in \mathcal{S}$ and $a \in K$. Then this map i satisfies the following conditions:

1) $i(X, a+b) = i(X, a) + i(X, b)$,

2) $i(X+Y, a) = i(X, a) + i(Y, a)$ if $XY=0$,

for each $X, Y \in S$, and $a, b \in K$. Further $\mathcal{J}$ is an extension of i.

Conversely, when such a map i is given, how can we extend the map i to an abstract integral $\mathcal{J}$? We shall give an answer to this question in the present part of the paper.

2. Construction of an abstract integral.

Assumption 1. *Let $(\mathcal{S}, \mathcal{F}, J)$ be an abstract integral structure and K a subgroup of $\mathcal{F}$. Let i be a map of $\mathcal{S} \times K$ into J satisfying the conditions:*

1) $i(X, a+b) = i(X, a) + i(X, b)$,

2) $i(X+Y, a) = i(X, a) + i(Y, a)$ if $XY=0$,

for each $X, Y \in \mathcal{S}$, and $a, b \in K$. Denote by $\mathcal{G}_0$ the subgroup of $\mathcal{F}$ generated by $\mathcal{S}K = \{Xa \mid X \in \mathcal{S}$ and $a \in K\}$ and by $\mathcal{G}$ the $\mathcal{F}$-completion [1] *of $\mathcal{G}_0$.*

Proposition 1. $\mathcal{G}_0 = \{\sum_{i=1}^{n} X_i a_i \mid X_i \in \mathcal{S}$ *and* $a_i \in K$, $i=1, 2, \cdots, n\}$ $= \{\sum_{i=1}^{n} X_i a_i \mid X_i \in \mathcal{S}$ *and* $a_i \in K$, $i=1, 2, \cdots, n$, *and* $X_j X_k = 0$ $(j \neq k)\}$.

Proof. It suffices to show that, for any $g = \sum_{i=1}^{n} X_i a_i \in \mathcal{G}_0$, where $X_i \in \mathcal{S}$ and $a_i \in K$, $i=1, 2, \cdots, n$, there exist $Y_j \in \mathcal{S}$ and $b_j \in K$,

$j=1, 2, \cdots, m$, such that $Y_j Y_{j'}=0$ $(j\neq j')$ and $g=\sum_{j=1}^{m} Y_j b_j$. This will be proved by induction on n. Since we have nothing to prove for $n=1$, it suffices to show, under the assumption that our assertion is true for $n=r-1$, that for $n=r$ there exist Y_j's and b_j's stated above. Our assumption implies that there exist $Z_k \in \mathcal{S}$ and $c_k \in K$, $k=1, 2, \cdots, l$, such that $Z_k Z_{k'}=0$ $(k\neq k')$ and $\sum_{i=1}^{r-1} X_i a_i=\sum_{k=1}^{l} Z_k c_k$. Put $Y_j=X_r Z_j$, $b_j=c_j+a_r$, $j=1, 2, \cdots, l$, put $Y_{l+j}=Z_j+Y_j$, $b_{l+j}=c_j$, $j=1, 2, \cdots, l$, and put $Y_{2l+1}=X_r+X_r \sum_{k=1}^{l} Z_k$, $b_{2l+1}=a_r$. Then it is easy to see that $Y_j \in \mathcal{S}$, $b_j \in K$, $j=1, 2, \cdots, 2l+1$, $Y_j Y_{j'}=0$ $(j\neq j')$ and that $\sum_{j=1}^{2l+1} Y_j b_j=\sum_{k=1}^{l} Z_k c_k+X_r a_r=\sum_{i=1}^{r} X_i a_i=g$. This completes the induction and thus Proposition 1 is proved.

Corollary. *$\mathcal{G}_0$ is an $\mathcal{S}$-invariant subgroup of $\mathcal{F}$.*

The corollary assures us that the $\mathcal{F}$-completion $\mathcal{G}$ of $\mathcal{G}_0$ is well defined. Further we have

Proposition 2. *K is contained in $\mathcal{G}$.*

The purpose of this part of the paper is to prove, under some assumptions, that the map i is uniquely extended to an abstract integral with respect to $(\mathcal{S}, \mathcal{G}, J)$.

First we shall show the uniqueness:

Proposition 3. *If the map i is extended to an abstract integral with respect to $(\mathcal{S}, \mathcal{G}, J)$, then such an abstract integral is uniquely determined.*

Proof. For $X \in \mathcal{S}$ and $g \in \mathcal{G}$, we have $Xg \in \mathcal{G}_0$ and hence there exist $X_i \in \mathcal{S}$ and $a_i \in K$, $i=1, 2, \cdots, n$, such that $Xg=\sum_{i=1}^{n} X_i a_i$. Thus we have $\mathcal{J}(X, g)=\mathcal{J}(X^2, g)=\mathcal{J}(X, Xg)=\mathcal{J}(X, \sum_{i=1}^{n} X_i a_i)=\sum_{i=1}^{n} \mathcal{J}(X, X_i a_i)$ $=\sum_{i=1}^{n} \mathcal{J}(XX_i, a_i)=\sum_{i=1}^{n} i(XX_i, a_i)$, and this proves the proposition.

To prove the existence of an abstract integral which is an extension of i, let us begin with a lemma which is easily verified.

Lemma 1. *If $X_i \in \mathcal{S}$, $i=1, 2, \cdots, m$, $X_i X_{i'}=0$ $(i\neq i')$ and if $Y_j \in \mathcal{S}$, $j=1, 2, \cdots, n$, $Y_j Y_{j'}=0$ $(j\neq j')$, then there exist $Z_{ij} \in \mathcal{S}$, $i=0, 1, \cdots, m$; $j=0, 1, \cdots, n$ $((i, j)\neq(0, 0))$, such that $Z_{ij} Z_{i'j'}=0$ $((i, j)\neq(i', j'))$, $X_i=\sum_{j=0}^{n} Z_{ij}$, $i=1, 2, \cdots, m$, and $Y_j=\sum_{i=0}^{m} Z_{ij}$, $j=1, 2, \cdots, n$. Moreover, these Z_{ij}'s are uniquely determined, respectively, as follows: $Z_{ij}=X_i Y_j$, $Z_{i0}=X_i+X_i \sum_{k=1}^{n} Y_k$ and $Z_{0j}=Y_j+Y_j \sum_{k=1}^{m} X_k$ for $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, n$.*

Corollary. *For any g and h in $\mathcal{G}_0$, there exist $X_i \in \mathcal{S}$, $a_i \in K$, and $b_i \in K$, $i=1, 2, \cdots, n$, such that $X_j X_k = 0$ $(j \neq k)$, $g = \sum_{i=1}^{n} X_i a_i$, and $h = \sum_{i=1}^{n} X_i b_i$.*

Under the following assumption, we shall show that the map i can be extended to an abstract integral with respect to $(\mathcal{S}, \mathcal{G}, J)$, except for the topological condition (in other words, if $\mathcal{G}$ is a discrete group).

Assumption 2. *If $X \in \mathcal{S}$, $X \neq 0$, $a \in K$, $a \neq 0$, then $Xa \neq 0$.*

Lemma 2. *If $X_i \in \mathcal{S}$, $a_i \in K$, $i=1, 2, \cdots, m$, $X_i X_{i'} = 0$ $(i \neq i')$, if $Y_j \in \mathcal{S}$, $b_j \in K$, $j=1, 2, \cdots, n$, $Y_j Y_{j'} = 0$ $(j \neq j')$, and if $\sum_{i=1}^{m} X_i a_i = \sum_{j=1}^{n} Y_j b_j$, then, for each i and j, it holds that*

1) $a_i = b_j$ *if* $X_i Y_j \neq 0$,

2) $a_i = 0$ *if* $X_i + X_i \sum_{k=1}^{n} Y_k \neq 0$,

3) $b_j = 0$ *if* $Y_j + Y_j \sum_{k=1}^{m} X_k \neq 0$.

Proof. Since $0 = X_i Y_j 0 = X_i Y_j (\sum_{r=1}^{m} X_r a_r - \sum_{s=1}^{n} Y_s b_s) = X_i Y_j a_i - X_i Y_j b_j = X_i Y_j (a_i - b_j)$, Assumption 2 implies 1). 2) follows from $0 = (X_i + X_i \sum_{k=1}^{n} Y_k)(\sum_{r=1}^{m} X_r a_r - \sum_{s=1}^{n} Y_s b_s) = (X_i + X_i \sum_{k=1}^{n} Y_k) a_i - \sum_{s=1}^{m} (X_i Y_s + X_i Y_s) b_s = (X_i + X_i \sum_{k=1}^{n} Y_k) a_i$ and 3) is proved in an analogous way.

Lemma 3. *There exists a unique homomorphism I of $\mathcal{G}_0$ into J such that*

$$I(Xa) = i(X, a) \text{ for each } X \in \mathcal{S} \text{ and } a \in K.$$

Proof. For any $g \in \mathcal{G}_0$ there exist $X_i \in \mathcal{S}$ and $a_i \in K$, $i=1, 2, \cdots \cdots, m$, such that $X_i X_{i'} = 0 (i \neq i')$ and $g = \sum_{i=1}^{m} X_i a_i$. The uniqueness of I follows from $I(g) = I(\sum_{i=1}^{m} X_i a_i) = \sum_{i=1}^{m} I(X_i a_i) = \sum_{i=1}^{m} i(X_i, a_i)$ and the existence is proved as follows. For another expression of g : $g = \sum_{j=1}^{n} Y_j b_j$, where $Y_j \in \mathcal{S}$, $b_j \in K$, $j=1, 2, \cdots, n$, and $Y_j Y_{j'} = 0$ $(j \neq j')$, we show that $\sum_{i=1}^{m} i(X_i, a_i) = \sum_{j=1}^{n} i(Y_j, b_j)$. For these X_i's and Y_j's, there exist $Z_{ij} \in \mathcal{S}$, for $i=0, 1, \cdots, m$ and $j=0, 1, \cdots, n$ $((i, j) \neq (0, 0))$, satisfying the conditions in Lemma 1. Lemma 2 implies that $a_i = b_j$ for $i \geqq 1$ and $j \geqq 1$ such that $Z_{ij} \neq 0$, that $a_i = 0$ for $i \geqq 1$ such that $Z_{i0} \neq 0$ and that $b_j = 0$ for $j \geqq 1$ such that $Z_{0j} \neq 0$. Thus we have $\sum_{i=1}^{m} i(X_i, a_i) = \sum_{i=1}^{m} i(\sum_{j=0}^{n} Z_{ij}, a_i) = \sum_{i=1}^{m} \sum_{j=0}^{n} i(Z_{ij}, a_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} i(Z_{ij}, a_i) = \sum_{j=1}^{n} \sum_{i=1}^{m} i(Z_{ij}, b_j) = \sum_{j=1}^{n} i(Y_j, b_j)$. Hence, for $g=\sum_{i=1}^{m} X_i a_i$, we can define $I(g)$ as $\sum_{i=1}^{m} i(X_i, a_i)$ unambiguously and thus a map I of $\mathcal{G}_0$ into J is defined. That the map I is a homomorphism is shown as follows. For g and h in $\mathcal{G}_0$, Corollary to Lemma 1 implies that there exist $X_i \in \mathcal{S}$, $a_i \in K$, and $b_i \in K$, $i=1, 2, \cdots \cdots, n$, such that $X_j X_k = 0$ $(j \neq k)$, $g=\sum_{i=1}^{n} X_i a_i$ and $h=\sum_{i=1}^{n} X_i b_i$. Then we have $I(g+h)=I(\sum_{i=1}^{n} X_i a_i + \sum_{i=1}^{n} X_i b_i)=I(\sum_{i=1}^{n} X_i(a_i+b_i))=\sum_{i=1}^{n} i(X_i, a_i+b_i) =\sum_{i=1}^{n} i(X_i, a_i)+\sum_{i=1}^{n} i(X_i, b_i)=I(g)+I(h)$. For $X \in \mathcal{S}$ and $a \in K$, that $I(Xa)=i(X, a)$ is obvious from the definition of I and this completes the proof of Lemma 3.

For an abstract integral structure $(\mathcal{S}, \mathcal{F}, J)$, a map $\mathcal{J}$ of $\mathcal{S}\times\mathcal{F}$ into J is called a *discrete abstract integral* with respect to the structure if it satisfies the conditions:

(*′) The map $\mathcal{J}=\mathcal{J}(X, f)$ is a homomorphism of $\mathcal{F}$ into J with respect to f for any fixed X.

(**) $\mathcal{J}(XY, f)=\mathcal{J}(X, Yf)$ for each $X, Y \in \mathcal{S}$, and $f \in \mathcal{F}$.

Any abstract integral is a discrete abstract integral and, conversely, a discrete abstract integral $\mathcal{J}$ is an abstract integral if and only if it satisfies the condition:

(*″) The map $\mathcal{J}=\mathcal{J}(X, f)$ is continuous with respect to f for any fixed X.

Now we can prove the following

Proposition 4. *The map i is uniquely extended to a discrete abstract integral $\mathcal{J}$ with respect to $(\mathcal{S}, \mathcal{G}, J)$.*

Proof. Define a map $\mathcal{J}$ of $\mathcal{S}\times\mathcal{G}$ into J by $\mathcal{J}(X, g)=I(Xg)$, for each $X \in \mathcal{S}$ and $g \in \mathcal{G}$, where I is the map in Lemma 3. Then it is easy to verify that the map $\mathcal{J}$ is a discrete abstract integral with respect to $(\mathcal{S}, \mathcal{G}, J)$ which is an extension of i. The uniqueness of such an extension follows from Proposition 3 when we consider $\mathcal{F}$ to be a discrete group and this completes the proof.

We see that a necessary and sufficient condition for the map i to be extended to an abstract integral with respect to $(\mathcal{S}, \mathcal{G}, J)$ is that the discrete abstract integral $\mathcal{J}$ in Proposition 4 satisfy Condition (*″) above.

It will be seen that a sufficient condition for (*″) is that the following Assumptions 3 and 4 be satisfied.

Assumption 3. *For any neighbourhood P of the unit element of $\mathcal{F}$, there exists a neighbourhood Q of the unit element of $\mathcal{F}$ such that*

$$f \in Q,\ a \in K,\ X \in \mathcal{S},\ X \neq 0, \text{ and } X(f-a)=0 \text{ imply } a \in P.$$

Assumption 4. *For any X in $\mathcal{S}$ and for any neighbourhood V of* *the unit element of J, there exists a neighbourhood P of the unit element of $\mathcal{F}$ such that*

$$a_i \in P \cap K,\ X_i \in \mathcal{S},\ i=1, 2, \cdots, n, \text{ and } X_j X_k = 0\ (j \neq k)$$
$$\text{imply } \sum_{i=1}^{n} i(XX_i, a_i) \in V.$$

Theorem 1. *Under Assumptions 1, 2, 3, and 4, the map i is uniquely extended to an abstract integral $\mathcal{J}$ with respect to the abstract integral structure $(\mathcal{S}, \mathcal{G}, J)$.*

Proof. The uniqueness has been proved in Proposition 3. Let $\mathcal{J}$ be the discrete abstract integral in Proposition 4. Then we need only prove that the map $\mathcal{J}$ satisfies Condition $(*'')$ above. Suppose $X \in \mathcal{S}$ and let V be any neighbourhood of the unit element of J. Then there exists a neighbourhood P of the unit element of $\mathcal{F}$ satisfying the condition in Assumption 4. For this neighbourhood P, there exists a neighbourhood Q of the unit element of $\mathcal{F}$ satisfying the condition in Assumption 3. Now, for given $g \in Q \cap \mathcal{G}$, we assert that $\mathcal{J}(X, g) \in V$, which proves the theorem. Since $Xg \in \mathcal{G}_0$, there exist $X_i \in \mathcal{S}$ and $a_i \in K$, $i=1, 2, \cdots, n$, such that $X_j X_k = 0$ $(j \neq k)$ and $Xg = \sum_{i=1}^{n} X_i a_i$. We may assume that $XX_i \neq 0$ for $1 \leqq i \leqq m$ and $XX_i = 0$ for $m < i \leqq n$, where m is an integer such that $0 \leqq m \leqq n$. Then, for each i, it holds that $XX_i(g - a_i) = XX_i Xg - XX_i a_i = XX_i \sum_{j=1}^{n} X_j a_j - XX_i a_i = XX_i a_i - XX_i a_i = 0$, which, by the definition of Q, implies that $a_i \in P$ for $1 \leqq i \leqq m$. Thus, by the definition of P, we have $\sum_{i=1}^{m} i(XX_i, a_i) \in V$. Hence $\mathcal{J}(X, g) = \mathcal{J}(X, Xg) = \mathcal{J}(X, \sum_{i=1}^{n} X_i a_i) = \sum_{i=1}^{n} \mathcal{J}(X, X_i a_i) = \sum_{i=1}^{n} \mathcal{J}(XX_i, a_i) = \sum_{i=1}^{n} i(XX_i, a_i) = \sum_{i=1}^{m} i(XX_i, a_i) \in V$. This completes the proof.

References

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[2] ——: Integration with respect to the generalized measure. II. Proc. Japan Acad., **43**, 184–185 (1967).