168. A Three Series Theorem

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(Comm. by Zyoiti SUETUNA, M. J. A., Oct. 12, 1968)

1. We suppose throughout this paper that (m_n) tends to zero monotonically.

J. Meder [1] (cf. S. Kaczmarz [2]) has proved the following

Theorem I. *Denote by l_n, L_n, and $\tilde{L}_n$ the first logarithmic means of the three series*

$$(1) \qquad \sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} a_n m_n, \quad \text{and} \quad \sum_{n=1}^{\infty} t_n \cdot \Delta^2 m_n$$

respectively, where $t_n = s_1 + s_2 + \cdots + s_n$ and $s_n = a_1 + a_2 + \cdots + a_n$. If

$$l_n = o(1/m_n) \qquad as \qquad n \to \infty$$

and

$$\Delta m_n = O(m_n / n \log n) \qquad as \qquad n \to \infty,$$

then $L_n = \tilde{L}_n + o(1)$ as $n \to \infty$.

He raized the problem ([1] P 471) whether this theorem holds also without any additional restriction or not and the problem ([1] P 472) to generalize this theorem by proving it e.g. in the case of weighted means or in the case of the Nörlund method of summation.

Let $p_n \geqq 0$, $p_1 > 0$, and $P_n = p_1 + p_2 + \cdots + p_n \to \infty$ as $n \to \infty$. The weighted mean w_n of the first series of (1) is defined by

$$w_n = (p_1 s_1 + p_2 s_2 + \cdots + p_n s_n)/P_n.$$

Similarly we denote by W_n and $\tilde{W}_n$ the weighted means of the second and the third series of (1).

The case $p_n = 1/n$ is the first logarithmic mean. About the weighted means J. Meder and Z. Zdrojewski [3] proved the following

Theorem II. *Suppose that $p_n > 0$, (p_n) is convex or concave and*

$$(2) \qquad 0 < \liminf_{n\to\infty} (n+1) p_n / P_n \leqq \limsup_{n\to\infty} (n+1) p_n / P_n < \infty.$$

If

$$(3) \qquad w_n = o(m_n^{-1}) \qquad as \qquad n \to \infty$$

and

$$(4) \qquad \Delta m_n = O(m_n / n) \qquad as \qquad n \to \infty,$$

then $W_n = \tilde{W}_n + o(1)$ as $n \to \infty$.

This theorem does not contain Theorem I as a particular case, since the first logarithmic means do not satisfy the condition (2). We shall prove the following

Theorem. *Suppose that*

(5) $\quad p_n \downarrow 0, \quad p_n/p_{n+1}=O(1) \quad$ *and* $\quad P_n \to \infty \quad$ *as* $\quad n\to\infty.$

In order that $W_n=\tilde{W}_n+o(1)$ *for all* (w_n) *satisfying the condition* (3) *it is necessary and sufficient that*

(6) $$\sum_{k=1}^{n} P_k m_k^{-1} \Delta m_{k+1} = O(P_n) \qquad as \qquad n\to\infty.$$

Since the first logarithmic means satisfy the condition (5), this theorem gives the solution for the problem P 471 and gives also the solution of P 472 in the case of weighted means. The case $p_n \downarrow 0$ in Theorem II is a particular case of this Theorem. For the case $p_n \downarrow 0$ we can find a necessary and sufficient condition from the proof of this Theorem, but it is not so simple as (6).

2. Proof of the Theorem. By the definition and Abel's lemma,

$$\begin{aligned}
P_nW_n &= \sum_{k=1}^{n} a_k m_k \sum_{j=k}^{n} p_j = \sum_{k=1}^{n} (s_k - s_{k-1}) m_k \sum_{j=k}^{n} p_j \\
&= \sum_{k=1}^{n-1} s_k \left(m_k \sum_{j=k}^{n} p_j - m_{k+1} \sum_{j=k+1}^{n} p_j \right) + s_n m_n p_n \\
&= \sum_{k=1}^{n-1} (t_k - t_{k-1}) \left(m_k \sum_{j=k}^{n} p_j + m_{k+1} p_k \right) + (t_n - t_{n-1}) m_n p_n \\
&= \sum_{k=1}^{n-2} t_k \left(\Delta^2 m_k \sum_{j=k}^{n} p_j + \Delta m_{k+1} \cdot p_k + \Delta(m_{k+1}p_k) \right) \\
&\quad + t_{n-1}(\Delta m_{n-1} \cdot (p_{n-1}+p_n) + m_n \Delta p_{n-1}) + t_n m_n p_n.
\end{aligned}$$

Therefore

(7) $$\begin{aligned}
P_nW_n &= P_n\tilde{W}_n + \sum_{k=1}^{n-2} t_k (2\Delta m_{k+1} \cdot p_k + \Delta p_k \cdot m_{k+2}) \\
&\quad + t_{n-1}(\Delta m_{n-1} \cdot (p_{n-1}+p_n) + m_n \Delta p_{n-1} - \Delta^2 m_{n-1} \cdot (p_{n-1}+p_n)) \\
&\quad + t_n (m_n p_n - \Delta^2 m_n \cdot p_n) \\
&= P_n\tilde{W}_n + X_n + Y_n + Z_n.
\end{aligned}$$

Now

$$P_n w_n = \sum_{k=1}^{n} a_k \sum_{j=k}^{n} p_j = \sum_{k=1}^{n} s_k p_k$$

and then

(8) $$\begin{aligned}
s_n &= p_n^{-1}(P_n w_n - P_{n-1} w_{n-1}), \\
t_n &= \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} p_k^{-1}(P_k w_k - P_{k-1} w_{k-1}) \\
&= P_n w_n p_n^{-1} + \sum_{k=1}^{n-1} P_k w_k \Delta(p_k^{-1}).
\end{aligned}$$

Substituting (8) into (7), we get

$$\begin{aligned}
X_n &= \sum_{k=1}^{n-2} \left(P_k w_k p_k^{-1} + \sum_{j=1}^{k-1} P_j w_j \Delta(p_j^{-1}) \right) (2p_k \Delta m_{k+1} + m_{k+2} \Delta p_k) \\
&= \sum_{k=1}^{n-2} P_k w_k p_k^{-1} (2p_k \Delta m_{k+1} + m_{k+2} \Delta p_k) \\
&\quad + \sum_{k=2}^{n-2} (2p_k \Delta m_{k+1} + m_{k+2} \Delta p_k) \sum_{j=1}^{k-1} P_j w_j \Delta(p_j^{-1})
\end{aligned}$$

$$=\sum_{k=1}^{n-2}P_kw_kp_k^{-1}(2p_k\Delta m_{k+1}+m_{k+2}\Delta p_k)$$
$$+\sum_{j=1}^{n-3}P_jw_j\Delta(p_j^{-1})\sum_{k=j+1}^{n-2}(2p_k\Delta m_{k+1}+m_{k+2}\Delta p_k)$$
$$=\sum_{k=1}^{n-2}P_kw_k\Big(2\Delta m_{k+1}+m_{k+2}p_k^{-1}\Delta p_k$$
$$+2\Delta(p_k^{-1})\sum_{j=k+1}^{n-2}p_j\Delta m_{j+1}+\Delta(p_k^{-1})\sum_{j=k+1}^{n-2}m_{j+2}\Delta p_j\Big).$$

Since we have

$$m_{k+2}p_k^{-1}\Delta p_k+\Delta(p_k^{-1})\sum_{j=k+1}^{n-2}m_{j+2}\Delta p_j$$
$$=p_k^{-1}\Delta p_k\Big(m_{k+2}-p_{k+1}^{-1}\sum_{j=k+1}^{n-2}m_{j+2}\Delta p_j\Big)$$
$$=p_k^{-1}\Delta p_k\Big(\Delta m_{k+2}+p_{k+1}^{-1}\sum_{j=k+2}^{n-2}p_j\Delta m_{j+1}+p_{n-1}m_n)\Big)$$
$$=-\Delta(p_k^{-1})\sum_{j=k+1}^{n-2}p_j\Delta m_{j+1}+p_k^{-1}\Delta p_k\cdot p_{n-1}m_n,$$

we get

$$(9)\qquad X_n=\sum_{k=1}^{n-2}P_kw_k\Big(2\Delta m_{k+1}+\Delta(p_k^{-1})\sum_{j=k+1}^{n-2}p_j\Delta m_{j+1}+p_k^{-1}\Delta p_k\cdot p_{n-1}m_n\Big).$$

Similarly,

$$Y_n=t_{n-1}((p_{n-1}+p_n)\Delta m_n+m_n\Delta p_{n-1})$$
$$=\Big(P_nw_np_n^{-1}+\sum_{k=1}^{n-1}P_kw_k\Delta(p_k^{-1})\Big)\Big((p_{n-1}+p_n)\Delta m_n+m_n\Delta p_{n-1}\Big)$$

and

$$Z_n=t_n(p_nm_n-p_n\Delta^2m_n)$$
$$=(p_nm_n-p_n\Delta^2m_n)\Big(P_nw_np_n^{-1}+\sum_{k=1}^{n-1}P_kw_k\Delta(p_k^{-1})\Big).$$

By the condition (3), the last term of X_n is

$$(10)\qquad p_{n-1}m_n\sum_{k=1}^{n-2}P_kw_kp_k^{-1}\Delta p_k=o\Big(P_n\sum_{k=1}^{n-2}\Delta p_k\Big)=o(P_n),$$

since $p_k^{-1}\uparrow$ and $m_k^{-1}\uparrow$, and

$$(11)\qquad Y_n=o\Big(P_nm_n^{-1}\Delta m_n+p_nP_n\sum_{k=1}^{n-1}|\Delta(p_k^{-1})|\Big)=o(P_n)$$

since $p_{n-1}/p_n=O(1)$, and further,

$$(12)\qquad Z_n=o\Big(P_n+p_nP_n\sum_{k=1}^{n-1}\Delta(p_k^{-1})\Big)=o(P_n).$$

Collecting (7), (9), (10), (11), and (12), we get

$$(13)\qquad P_nW_n=P_n\tilde{W}_n+\sum_{k=1}^{n-2}P_kw_k\Big(2\Delta m_{k+1}+\Delta(p_k^{-1})\sum_{j=k+1}^{n-2}p_j\Delta m_{j+1}\Big)$$
$$+o(P_n).$$

Now

$$(14)\qquad 0\leqq-\sum_{k=1}^{n-2}P_km_k^{-1}\Delta(p_k^{-1})\sum_{j=k+1}^{n-2}p_j\Delta m_{j+1}$$
$$=-\sum_{j=2}^{n-2}p_j\Delta m_{j+1}\sum_{k=1}^{j-1}P_km_k^{-1}\Delta(p_k^{-1})$$

$$\leqq \sum_{j=2}^{n-2} p_j \Delta m_{j+1} \cdot P_{j-1} m_{j-1}^{-1} \sum_{k=1}^{j-1} |\Delta(p_k^{-1})|$$
$$\leqq \sum_{j=2}^{n-2} P_{j-1} m_{j-1}^{-1} \Delta m_{j+1} \leqq \sum_{j=2}^{n-2} P_j m_j^{-1} \Delta m_{j+1}.$$

Therefore (13) becomes

$$P_n W_n = P_n \tilde{W}_n + o\left(\sum_{k=1}^{n-2} P_k m_k^{-1} \Delta m_{k+1}\right) + o(1).$$

This proves the sufficiency of the condition (6). The necessity of the condition (6) is seen by (13) and (14). Thus the Theorem is proved.

References

[1] J. Meder: On a lemma of S. Kaczmarz. Colloquium Mathematicum, **12**, 253–258 (1964).

[2] S. Kaczmarz: Sur la convergence et sommabilité des développements orthogonaux. Studia Mathematica, **1**, 87–121 (1929), Lemma 5 (p. 111).

[3] J. Meder and Z. Zdrojewski: On a relation between some special methods of summation. Colloquium Mathematicum, **19**, 131–142 (1968).