194. On Free Contents

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1. Introduction. An S-indecomposable semigroup is a semigroup which has no semilattice-homomorphic image except a trivial one. We will call an S-indecomposable semigroup $\mathfrak{P}$-simple in the sense that a semigroup S is S-indecomposable if and only if it has no prime ideal, that is, S has no ideal I such that $I \neq S$ and $S \setminus I$ is a subsemigroup of S (cf. [1]).

Let S be a semigroup. Let $a_1, \cdots, a_n$ be a finite number of elements of S. All the elements x of S each of which is the product of all of $a_1, \cdots, a_n$ (admitting repeated use) form a subsemigroup of S. It is denoted by $C_S(a_1, \cdots, a_n)$ or C_S and is called the content of $a_1, \cdots, a_n$ is S. We notice that $a_1, \cdots, a_n$ need not be distinct. For example, however, $C_S(a)$ is different from $C_S(a, a)$ in general: $C_S(a) = \{a^i ; i \geqq 1\}$ but $C_S(a, a) = \{a^i ; i \geqq 2\}$. Let F_n be the free semigroup generated by $a_1, \cdots, a_n$. Then $C_{F_n}(a_1, \cdots, a_n)$ is called the free content of $a_1, \cdots, a_n$. The author did not use the terminology "content" and "$\mathfrak{P}$-simplicity" in the preceding papers [2], [3] but he proved there

(1) A free content is $\mathfrak{P}$-simple.

(2) A content is $\mathfrak{P}$-simple.

(3) A semigroup is a semilattice-union of $\mathfrak{P}$-simple semigroups.

(4) In the greatest semilattice-decomposition (S-decomposition) of a semigroup, each congruence class is $\mathfrak{P}$-simple.

The author discussed these in the two ways: one way is along the direction, (4)→(3)→(1)→(2) after directly proving (4) [2]. The other way is along the direction, (1)→(2)→(4)→(3) after directly proving (1) [3]. The concept of content is important and interesting but its structure has not been studied so much. In this short note we report a few results on free contents. The detailed proof will be published elsewhere [4].

2. Rank. The positive number n of $C_{F_n}(a_1, \cdots, a_n)$ is called the rank of a free content C_{F_n}. For simplicity the free content of rank n is denoted by $\mathcal{F}_n$.

$$\mathcal{F}_n = C_{F_n}(a_1, \cdots, a_n).$$

The letters $a_1, \cdots, a_n$ are called the generators of $\mathcal{F}_n$, but they are not elements of $\mathcal{F}_n$. We have the following theorem.

Theorem 1. *$\mathcal{F}_m$ is isomorphic onto $\mathcal{F}_n$ if and only if $m=n$. The rank m of a free content $\mathcal{F}_m$ is the minimum of n's for which $\mathcal{F}_m$ can be embedded into a free semigroup F_n as a maximal $\mathfrak{P}$-simple subsemigroup.*

We observe some property of prime-factorization in a free content. The property is required to be invariant under isomorphism. Let $W \in \mathcal{F}_n = C_{F_n}(a_1, \cdots, a_n)$, $n>1$, and let $W = x_1x_2\cdots x_k$ where the set $\{x_1, \cdots, x_k\}$ is equal to the set $\{a_1, \cdots, a_n\}$. W is called a prime if $W \in \mathcal{F}_n$ but $\notin \mathcal{F}_n^2$. For $W = x_1x_2\cdots x_k$, define $\mathcal{L}(W) = x_1x_2\cdots x_l$, $l \leqq k$, where $\{x_1, \cdots, x_l\} = \{a_1, \cdots, a_n\}$ but $\{x_1, \cdots, x_{l-1}\} \neq \{a_1, \cdots, a_n\}$. Then $\mathcal{L}(W)$ is called the left main of W. Likewise the right main $\mathcal{R}(W)$ of W can be defined. W is called left (right) minimal if $W = \mathcal{L}(W)$ $(W = \mathcal{R}(W))$. W is called minimal if $\mathcal{L}(W) = \mathcal{R}(W) = W$. The k of $W = x_1\cdots x_k$ is denoted by $k = |W|$. W is called a permutation if $|W| = n$. Every element of $\mathcal{F}_n$ is the product of primes but the factorization need not be unique. W is uniquely factorizable if and only if W is either a prime or $W = W_1W_2$ where W_1 is left minimal and W_2 is right minimal. If W is factorized into the product of two primes then W is called two-prime factorizable. Then we have characterization of permutations:

Lemma. *W is a permutation in $\mathcal{F}_n$ if and only if W^2 is uniquely factorizable, W^3 is two-prime factorizable and the number of two-prime factorizations of W^3 is the minimum of the numbers of those two-prime factorizations of elements of the form X^3 where X are minimal.*

By using this lemma we can prove the former half of Theorem 1. The latter half is an immediate consequence.

We have other interesting results, Theorems 2, 3:

Theorem 2. *$\mathcal{F}_m$ is isomorphic into $\mathcal{F}_n$ if and only if $n>1$.*

Theorem 2 is equivalent to (5) and (6) below.

(5) $\mathcal{F}_m$ is isomorphic into $\mathcal{F}_{m+1}$.

(6) $\mathcal{F}_m$ is isomorphic into $\mathcal{F}_2$ if $m>2$.

Theorem 3. *If $m>n$, $\mathcal{F}_m$ is homomorphic onto $\mathcal{F}_n$.*

However the following question is still open.

Problem. If $m<n$, is $\mathcal{F}_m$ homomorphic onto $\mathcal{F}_n$?

3. Structure. Let S be a set and $\mathfrak{B}_S$ denote the set of all binary operations defined on S. The two binary operations a^* and $^*a$ are defined on $\mathfrak{B}_S$ for each $a \in S$ in the following way: For $\theta, \eta \in \mathfrak{B}_S$, x, $y \in S$.

$$x(\theta\ a^*\ \eta)y = (x\ \theta a)\eta y, \qquad x(\theta\ {}^*a\ \eta)y = x\theta(a\ \eta y).$$

Let T be a semigroup. Consider a mapping Θ of $T \times T$ into $\mathfrak{B}_S$:

$$(\alpha, \beta)\Theta = \theta_{\alpha,\beta}, \qquad (\alpha, \beta) \in T \times T$$

subject to

$$\theta_{\alpha,\beta}\, a^*\, \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\beta\gamma} {}^* a\, \theta_{\beta,\gamma} \qquad \text{for all } \alpha, \beta, \gamma \in T \quad \text{all } a \in S.$$

Given S, T, Θ, a binary operation is defined on $S \times T$ by

$$(x, \alpha)(y, \beta) = (x\theta_{\alpha,\beta} y, \alpha\beta). \tag{7}$$

The semigroup $S \times T$ with (7) is called a general product of a set S by a semigroup T with respect to Θ and it is denoted by $S \overline{\times}_\Theta T$ or $S \rtimes\!\!\!\ltimes T$.

Returning to free contents, let $\mathcal{F} = C_{F_n}(a_1, \cdots, a_n)$. For each $\alpha \in F_n^1 = F_n \cup \{1\}$ (1 is a void word), the two transformations φ_α and ψ_α of $\mathcal{F}$ are defined by $X\varphi_\alpha = X\alpha$, $\psi_\alpha X = \alpha X$, where $X \in \mathcal{F}$. Clearly $X(\alpha\beta) = (X\alpha)\beta$, $(\alpha\beta)X = \alpha(\beta X)$, $(\alpha X)\beta = \alpha(X\beta)$. Let $\mathcal{L}$ be the set of all left minimal elements of $\mathcal{F}$. Each $X \in \mathcal{F}$ has a unique expression

$$X = A\varphi_\alpha \text{ for some } A \in \mathcal{L},\ \alpha \in F_n^1.$$

Then we have

Theorem 4. *Let $\mathcal{F}$ be a free content and let $\mathcal{L}$ be the left zero semigroup defined on the set of all left minimal elements of $\mathcal{F}$. For each $A \in \mathcal{L}$ we define a binary operation θ_A on F_n^1 by*

$$\alpha\theta_A\beta = \alpha A\beta, \qquad \alpha, \beta \in F_n^1.$$

Let $\Theta = \{\theta_A ; A \in \mathcal{L}\}$. Then $\mathcal{F}$ is isomorphic onto $F_n^1 \overline{\times}_\Theta \mathcal{L}$, i.e., the set

$$F_n^1 \times \mathcal{L} = \{(\alpha, A) ; \alpha \in F_n^1, A \in \mathcal{L}\}$$

in which the operation is defined by

$$(\alpha, A)(\beta, B) = (\alpha\theta_B\beta, A).$$

However, the abstract characterization of a free content in terms of general product is still open. Finally we have the decomposition theory of free contents.

Let ξ_l and σ be the relations on a free content $\mathcal{F}$ defined as follows:

$$X\xi_l Y \text{ iff } \mathcal{L}(X) = \mathcal{L}(Y).$$
$$X\sigma Y \text{ iff } \mathcal{L}(X) = \mathcal{L}(Y) \text{ and } \mathcal{R}(X) = \mathcal{R}(Y).$$

Theorem 5. *ξ_l is the smallest left zero congruence on $\mathcal{F}$, and σ is the smallest idempotent congruence on $\mathcal{F}$.*

References

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