231. *Proofs of Some Axioms by Stroke Function*

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In this paper, we shall give proofs of some axioms written by the stroke function. For details of the stroke function and deduction rules, see R. Price [1].

The Rules of the System.

Stroke Introduction $(\mid I)$

1	$\quad p$	hypothesis
$\vdots$	$\quad \vdots$	
2	$\quad q \mid q$	assumption
3	$p \mid q$	$1-2, \mid 1$

Stroke Elimination $(\mid E)$

1	p	hypothesis
2	$q \mid p$	hypothesis
3	$q \mid q$	$1.2, \mid E$

Double stroke Elimination $(\| E)$

1	p	hypothesis
2	$(q \mid q) \mid p$	hypothesis
3	q	$1.2, \| E$

These three rules yield many other deduction schemata (see [1]). The followings are used in this paper.

Deduction schemata

Negation Introduction $(\sim I)$

1	$\quad p$	hypothesis
$\vdots$	$\quad \vdots$	
2	$\quad q$	assumption
3	$\quad q \mid q$	assumption
4	$p \mid p$	$1-3, \sim I$

Tautology (Taut)

1	$p \mid (p \mid p)$	
$\vdots$	$\vdots$	

Rules of Detachment (Nicord)

1	p	hypothesis
2	$p \mid (q \mid r)$	hypothesis
3	r	1.2, Nicord

Second Rules of Detachment (Docin)

1	p	hypothesis
2	$p \mid (q \mid r)$	hypothesis
3	q	1.2, Docin

Commutivity for Stroke ($\mid$ Comm)

1	$p \mid q$	hypothesis
2	$q \mid p$	1, Comm

We use the following abbreviations of terms:

H for hypothesis, D for Docin, N for Nicord, R for repetition, C for $\mid$ Comm.

We shall prove the following three axioms:

Axiom 1. $(p \mid (q \mid r)) \mid (s \mid (s \mid s)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s))))$ (Lukasiewicz)

Axiom 2. $(p \mid (q \mid r)) \mid (((s \mid r) \mid ((p \mid s) \mid (p \mid s))) \mid (p \mid (p \mid q)))$ (Wajsberg)

Axiom 3. $(p \mid (q \mid r)) \mid ((p \mid (r \mid p)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s))))$ (Lukasiewicz)

Proof of Axiom 1.

1	$(s \mid (s \mid s)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s)))$	H
2	$p \mid (q \mid r)$	H
3	$s \mid (s \mid s)$	Taut
4	$s \mid q$	3.1, D
5	p	H
6	q	5.2, D
7	$s \mid q$	4, R
8	$s \mid s$	6.7, $\mid E$
9	$p \mid s$	5–8, $\mid I$
10	$(p \mid s) \mid (p \mid s)$	3.1, N
11	$(p \mid (q \mid r)) \mid (p \mid (q \mid r))$	2–10, $\sim I$
12	$((s \mid (s \mid s) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s)))) \mid (p \mid (q \mid r))$	1–11, $\mid I$
13	$(p \mid (q \mid r)) \mid ((s \mid (s \mid s)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s))))$	12, C

Proof of Axiom 2.

1	$((s \mid r) \mid ((p \mid s) \mid (p \mid s))) \mid (p \mid (p \mid q))$	H
2	$p \mid (q \mid r)$	H
3	$(s \mid r) \mid ((p \mid s) \mid (p \mid s))$	H
4	p	3.1, D
5	q	4.2, D
6	$p \mid q$	3.1, N
7	$p \mid p$	5.6, $\mid E$
8	$((s \mid r) \mid ((p \mid s) \mid (p \mid s))) \mid ((s \mid r) \mid ((p \mid s) \mid (p \mid s)))$	3–7, $\sim I$
9	$((s \mid r) \mid ((p \mid s) \mid (p \mid s))) \mid ((s \mid r) \mid ((p \mid s) \mid (p \mid s)))$	H
10	$(s \mid r) \mid ((p \mid s) \mid (p \mid s))$	H
11	p	1.10, D
12	q	2.11, D
13	$p \mid q$	1.10, N
14	$p \mid p$	12.13, $\mid E$
15	$((s \mid r) \mid ((p \mid s) \mid (p \mid s))) \mid ((s \mid r) \mid ((p \mid s) \mid (p \mid s)))$	10–14, $\sim I$
16	$(((s \mid r) \mid ((p \mid s) \mid (p \mid s))) \mid ((s \mid r) \mid ((p \mid s) \mid (p \mid s)))) \mid ((s \mid r) \mid ((p \mid s) \mid (p \mid s)))$	9–15, $\mid I$

17	p	H
18	s	H
19	$s \mid r$	$8.16, D$
20	$r \mid s$	$19, C$
21	$r \mid r$	$18.20, \mid E$
22	r	$2.17, D$
23	$s \mid s$	$18\text{–}22, \sim I$
24	$p \mid s$	$17\text{–}23, \mid I$
25	$(p \mid (q \mid r)) \mid (p \mid (q \mid r))$	$8.16, N$
26	$(p \mid (q \mid r)) \mid (p \mid (q \mid r))$	$2\text{–}25, \sim I$
27	$(((s \mid r) \mid ((p \mid s) \mid (p \mid s))) \mid (p \mid q))) \mid (p \mid (q \mid r))$	$1\text{–}26, \mid I$
28	$(p \mid (q \mid r)) \mid (((s \mid r) \mid)(p \mid s) \mid (p \mid s))) \mid (p \mid (p \mid (p \mid q)))$	$27, C$

Proof of Axiom 3.

1	$(p \mid (r \mid p)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s)))$	H
2	$p \mid (q \mid r)$	H
3	$p \mid (r \mid p)$	H
4	$s \mid q$	$3.1, D$
5	$(p \mid s) \mid (p \mid s)$	$3.1, N$
6	p	H
7	q	$6.2, D$
8	$s \mid q$	$4, R$
9	$s \mid s$	$7.8, \mid E$
10	$p \mid s$	$6.9, \mid I$
11	$(p \mid s) \mid (p \mid s)$	$5, R$
12	$(p \mid (r \mid p)) \mid (p \mid (r \mid p))$	$3\text{–}11, \sim I$
13	$(p \mid (r \mid p) \mid (p \mid (r \mid p))$	H
14	$p \mid (r \mid p)$	H
15	p	$13.14, D$
16	$r \mid p$	$13.14, N$
17	$p \mid p$	$14.16, \mid E$
18	$(p \mid (r \mid p)) \mid (p \mid (r \mid p))$	$14\text{–}17, \sim I$
19	$((p \mid (r \mid p)) \mid (p \mid (r \mid p))) \mid (p \mid (r \mid p))$	$13\text{–}18, \mid I$
20	p	H
21	$r \mid p$	H
22	r	$2.20, N$
23	$p \mid r$	$21, C$
24	$p \mid p$	$22.23, \mid E$
25	$(r \mid p) \mid p$	$21\text{–}24, \mid I$
26	$p \mid (r \mid p)$	$25, C$
27	$(p \mid (r \mid p)) \mid (p \mid (r \mid p))$	$12, R$
28	$p \mid p$	$20\text{–}27, \mid I$
29	p	$12.19, D$

30	$\mid (p \mid (q \mid r)) \mid (p \mid (q \mid r))$	2–29, $\sim I$
31	$((p \mid (r \mid p)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s)))) \mid (p \mid (q \mid r))$	1–30, $\mid I$
32	$(p \mid (q \mid r)) \mid ((p \mid (r \mid p)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s))))$	31, C

Reference

[1] Robert Price: The Stroke Function in natural deduction. Zeitschr. f. math. Logik und Grundlagen d. Math., **7**, 117–123 (1961).