226. Relations between Unitary ρ-Dilatations and Two Norms. II

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1. Following [1] [4] [7] an operator T on a Hilbert space H possesses a unitary ρ-dilatation if there exist a Hilbert space K containing H as a subspace, a positive constant ρ and a unitary operator U on K satisfying the following representation

$$(1) \qquad T^n = \rho \cdot PU^n \quad (n=1, 2, \cdots\cdots)$$

where P is the orthogonal projection of K on H. Put $\mathcal{C}_\rho$ the class of all operators on H having a unitary ρ-dilatation on a suitable enlarged space K. These classes $\mathcal{C}_\rho$ $(\rho \geqq 0)$ were introduced by Sz-Nagy and C. Foias [7]. They have shown a characterization and the monotonity of $\mathcal{C}_\rho$. In the previous paper [4] we obtained the condition for the operator norm $\|T\|$ and the numerical radius $\|T\|_N$ satisfied by T in $\mathcal{C}_\rho$ $(\rho \leqq 2)$,

that is if $T \in \mathcal{C}_\rho$ $(0 \leqq \rho \leqq 1)$, *then*

$$1/2\|T\| \leqq \|T\|_N \leqq \begin{cases} \|T\| & \left(0 \leqq \|T\| \leqq \dfrac{\rho}{2-\rho}\right) \\ \dfrac{\rho}{2-\rho} & \left(\dfrac{\rho}{2-\rho} \leqq \|T\| \leqq \rho\right) \end{cases}$$

and if $T \in \mathcal{C}_\rho$ $(1 \leqq \rho \leqq 2)$, *then*

$$1/2\|T\| \leqq \|T\|_N \leqq \begin{cases} \|T\| & (0 \leqq \|T\| \leqq 1) \\ 1 & (1 \leqq \|T\| \leqq \rho). \end{cases}$$

In this paper we continue the investigation for classes $\mathcal{C}_\rho$ $(\rho \geqq 2)$. We give a simple necessary condition for $T \in \mathcal{C}_\rho$ $(\rho \geqq 2)$ related to both $\|T\|$ and $\|T\|_N$ and its graphic representation.

2. The following theorems are known and we cite for the sake of convenience ([2] [4] [7]).

Theorem A. *An operator T in H belongs to the class $\mathcal{C}_\rho$ if and only if it satisfies the following conditions*

$$(\text{i}) \begin{cases} (\text{I}_\rho) \quad \|h\|^2 - 2\left(1-\dfrac{1}{\rho}\right)\text{Re}(zTh, h) + \left(1-\dfrac{2}{\rho}\right)\|zTh\|^2 \geqq 0 \\ \qquad \textit{for } h \textit{ in } H \textit{ and } |z| \leqq 1, \\ (\text{II}) \quad \textit{the spectrum of } T \textit{ lies in the closed unit disk.} \end{cases}$$

(ii) *If $\rho \leqq 2$, then the condition (I_ρ) implies (II).*

Using the notion of shell, Ch. Davis [2] has proved the following proposition.

Proposition. *If $\rho\geqq 2$, then the condition (I_ρ) also implies (II).* This proposition was implicitly contained in [7]. Thus we have the following theorem.

Theorem A′. *An operator T belongs to C_ρ if and only if it satisfies the condition (I_ρ).*

Theorem B. *C_ρ is non-decreasing with respect to the index ρ in the sense that*

$$C_{\rho_1}\subset C_{\rho_2} \quad \text{if } 0<\rho_1\leqq\rho_2.$$

The following theorems were proved in [4].

Theorem C. (i) *If $T\in C_\rho$ for $0\leqq\rho\leqq 1$, then $\|T\|_N\leqq\frac{\rho}{2-\rho}$.*

(ii) *If $T\in C_\rho$ for $1\leqq\rho\leqq 2$, then $\|T\|_N\leqq 1$.*

(iii) *If $(2-\rho)\|T\|^2+2(1-\rho)\|T\|_N-\rho\leqq 0$ for $0\leqq\rho\leqq 1$, then $T\in C_\rho$.*

(iv) *If $(2-\rho)\|T\|^2+2(\rho-1)\|T\|_N-\rho\leqq 0$ for $1\leqq\rho\leqq 2$, then $T\in C_\rho$.*

Theorem D. (i) *If $T\in C_\rho$, there exists k in $[1/2,1]$ such that*

$$(2-\rho)\|T\|^2k^2+2(1-\rho)\|T\|_N-\rho\leqq 0 \quad \text{for } 0\leqq\rho\leqq 1.$$

(ii) *If $T\in C_\rho$, there exists k in $[1/2,1]$ such that*

$$(2-\rho)\|T\|^2k^2+2(\rho-1)\|T\|_N-\rho\leqq 0 \quad \text{for } 1\leqq\rho\leqq 2.$$

3. For $2\leqq\rho$, the condition (I_ρ) is replaced by

$$(\rho-2)\|zTh\|^2-2(\rho-1)|(Th,h)|r\cos\psi+\rho\|h\|^2\geqq 0 \text{ for } h \text{ in } H,\ |z|\leqq 1$$

that is

$$(I'_\rho) \qquad (\rho-2)\|Th\|^2r^2-2(\rho-1)|(Th,h)|r\cos\psi+\rho\geqq 0$$

for every unit vector h in H, where $z=re^{i\theta}$, $0\leqq r\leqq 1$, $\psi=\varphi+\theta$ and φ is the argument of (Th,h).

Since the left hand side of (I_ρ) is positive if it is so when $\cos\psi=1$, (I'_ρ) is equivalent to

$$(I''_\rho) \qquad (\rho-2)\|Th\|^2r^2-2(\rho-1)|(Th,h)|r+\rho\geqq 0$$

for every unit vector h in H and for $0\leqq r\leqq 1$.

Lemma. *If $T\in C_\rho$ for $\rho\geqq 2$, there exists k in $[1/2,1]$ such that*

$$(\rho-2)\|T\|^2k^2r^2-2(\rho-1)\|T\|_Nr+\rho\geqq 0 \quad \text{for } 0\leqq r\leqq 1.$$

Proof. Let $\{h_n\}$ be a sequence of unit vectors which $|(Th_n,h_n)|$ converges to $\|T\|_N$. Then

$$|(Th_n,h_n)|\leqq\|Th_n\|\leqq\|T\|,$$

hence

$$\|T\|_N\leqq\sup\|Th_n\|\leqq\|T\|.$$

Thus we get

$$\frac{1}{2}\leqq\frac{\|T\|_N}{\|T\|}\leqq\frac{\sup\|Th_n\|}{\|T\|}\leqq 1.$$

Put $k=\frac{\sup\|Th_n\|}{\|T\|}$, then $1/2\leqq k\leqq 1$ and $\sup\|Th_n\|=k\|T\|$. By (I''_ρ) we have

$$(\rho-2)\|Th_n\|^2r^2-2(\rho-1)|(Th_n,h_n)|r+\rho\geqq 0 \quad \text{for } 0\leqq r\leqq 1,$$
$$(\rho-2)\|T\|^2k^2r^2-2(\rho-1)\|T\|_N\ r+\rho\geqq 0 \quad \text{for } 0\leqq r\leqq 1.$$

By Theorem A′, the proof is complete.

Theorem.

(i) *If* $T\in C_\rho$ *for* $2\leqq\rho\leqq\sqrt{2}+1$, *then*

$$1/2\|T\|\leqq\|T\|_N\leqq\begin{cases}\|T\| & (0\leqq\|T\|\leqq 1)\\ \dfrac{\rho-2}{2(\rho-1)}\|T\|^2+\dfrac{\rho}{2(\rho-1)} & (1\leqq\|T\|\leqq\rho).\end{cases}$$

(ii) *If* $T\in C_\rho$ *for* $\rho\geqq\sqrt{2}+1$, *then*

$$1/2\|T\|\leqq\|T\|_N\leqq\begin{cases}\|T\| & (0\leqq\|T\|\leqq 1)\\ \dfrac{\rho-2}{2(\rho-1)}\|T\|^2+\dfrac{\rho}{2(\rho-1)} & \left(1\leqq\|T\|\leqq\sqrt{\dfrac{\rho}{\rho-2}}\right)\\ \dfrac{\sqrt{\rho(\rho-2)}}{\rho-1}\|T\| & \left(\sqrt{\dfrac{\rho}{\rho-2}}\leqq\|T\|\leqq\rho\right).\end{cases}$$

Proof. We put

$$\mathcal{F}_{\rho,k,r}(\|T\|,\|T\|_N)\equiv(\rho-2)\|T\|^2k^2r^2-2(\rho-1)\|T\|_N r+\rho$$

and define the following domains in the $(\|T\|,\|T\|_N)$ plane

$$\mathcal{D}_{\rho,k,r}(\|T\|,\|T\|_N)\equiv\{(\|T\|,\|T\|_N)\,;\,\mathcal{F}_{\rho,k,r}(\|T\|,\|T\|_N)\geqq 0 \text{ for some } r\in[0,1]\}$$

$$\mathcal{D}_{\rho,k}(\|T\|,\|T\|_N)\equiv\bigcap_{0\leqq r\leqq 1}\mathcal{D}_{\rho,k,r}(\|T\|,\|T\|_N)$$

$$\mathcal{D}_{\rho}(\|T\|,\|T\|_N)\equiv\bigcup_{\frac{1}{2}\leqq k\leqq 1}\mathcal{D}_{\rho,k}(\|T\|,\|T\|_N).$$

Then by lemma the domain $\mathcal{D}_\rho(\|T\|,\|T\|_N)$ indicates the necessary condition for $T\in C_\rho$ in the sense that if $T\in C_\rho$, then $(\|T\|,\|T\|_N)\in\mathcal{D}_\rho(\|T\|,\|T\|_N)$.

Now let us consider the envelope of $\mathcal{F}_{\rho,k,r}(\|T\|,\|T\|_N)=0$ for all r and fixed ρ, k as follows. We eliminate the parameter r from the simultaneous equations

$$\begin{cases}\mathcal{F}_{\rho,k,r}(\|T\|,\|T\|_N)=0\\ \dfrac{\partial\mathcal{F}_{\rho,k,r}(\|T\|,\|T\|_N)}{\partial r}=0\end{cases}$$

then we get the line

$$\|T\|_N=\frac{k\sqrt{\rho(\rho-2)}}{\rho-1}\|T\|$$

as the envelope.

We define $\mathcal{D}_{E(\rho,k)}(\|T\|,\|T\|_N)$ and $D^L_{\rho,k,1}(\|T\|,\|T\|_N)$ by

$$\mathcal{D}_{E(\rho,k)}(\|T\|,\|T\|_N)\equiv\left\{(\|T\|,\|T\|_N)\,;\,\|T\|_N\leqq\frac{k\sqrt{\rho(\rho-2)}}{\rho-1}\|T\|\right\}$$

$$\mathcal{D}^L_{\rho,k,1}(\|T\|,\|T\|_N)\equiv\left\{\mathcal{D}_{\rho,k,1}(\|T\|,\|T\|_N)\,;\,\|T\|\leqq\frac{1}{k}\sqrt{\frac{\rho}{\rho-2}}\right\}.$$

Since the curve $\mathcal{F}_{\rho,k,1}(\|T\|,\|T\|_N)$ contacts the envelope of $\mathcal{F}_{\rho,k,r}(\|T\|,\|T\|_N)=0$ at $E'\left(\frac{1}{k}\sqrt{\frac{\rho}{\rho-2}},\ \frac{\rho}{\rho-1}\right)$, we have

$$\mathcal{D}_{\rho,k}(\|T\|,\|T\|_N)=\mathcal{D}^L_{\rho,k,1}(\|T\|,\|T\|_N)\cup\mathcal{D}_{E(\rho,k)}(\|T\|,\|T\|_N).$$

The slope of the envelope of $\mathcal{F}_{\rho,k,r}(\|T\|, \|T\|_N)=0$ is less than that of $\mathcal{F}_{\rho,1,r}(\|T\|, \|T\|_N)=0$ and the curve $\mathcal{F}_{\rho,k,1}(\|T\|, \|T\|_N)=0$ lies lower than the curve $\mathcal{F}_{\rho,1,1}(\|T\|, \|T\|_N)=0$. Hence we get

$$\mathcal{D}_{\rho,k}(\|T\|, \|T\|_N)\subset\mathcal{D}_{\rho,1}(\|T\|, \|T\|_N) \quad \text{for all } k\in[1/2,1]$$

consequently

$$\mathcal{D}_{\rho}(\|T\|, \|T\|_N)\equiv \bigcup_{\frac{1}{2}\leqq k\leqq 1} \mathcal{D}_{\rho,k}(\|T\|, \|T\|_N)=\mathcal{D}_{\rho,1}(\|T\|, \|T\|_N).$$

Hence if $\sqrt{\frac{\rho}{\rho-2}}\geqq\rho$ i.e., $2\leqq\rho\leqq\sqrt{2}+1$, $\mathcal{D}_{\rho}(\|T\|, \|T\|_N)$ is enclosed by the three lines $\|T\|_N=\|T\|$, $\|T\|_N=1/2\|T\|$, $\|T\|=\rho$ and the curve $\mathcal{F}_{\rho,1,1}(\|T\|, \|T\|_N)=0$ (see Fig. 2), if $\sqrt{\frac{\rho}{\rho-2}}\leqq\rho$, i.e., $\rho\geqq\sqrt{2}+1$, $\mathcal{D}_{\rho}(\|T\|, \|T\|_N)$ is enclosed by the four lines $\|T\|_N=\|T\|$, $\|T\|_N=1/2\|T\|$, $\|T\|=\rho$, the envelope $\|T\|_N=\frac{\sqrt{\rho(\rho-2)}}{\rho-1}\|T\|$, and the curve $\mathcal{F}_{\rho,1,1}(\|T\|, \|T\|_N)=0$ (see Fig. 1).

In Fig. 1 the curve AE (AD in Fig. 2) and the envelope line ED (DE in Fig. 2) are respectively given by

$$f_1(\rho)\ ;\ \|T\|_N=\frac{\rho-2}{2(\rho-1)}\|T\|^2+\frac{\rho}{2(\rho-1)}$$

$$f_E(\rho)\ ;\ \|T\|_N=\frac{\sqrt{\rho(\rho-2)}}{\rho-1}\|T\|.$$

$f_1(\rho)$ contacts $f_E(\rho)$ at $E\left(\sqrt{\frac{\rho}{\rho-2}}, \frac{\rho}{\rho-1}\right)$. Moreover when $\rho\to\infty$, $\frac{\rho-2}{2(\rho-1)}$ gradually tends to 1/2 and the slope of $f_E(\rho)$, $\frac{\sqrt{\rho(\rho-2)}}{\rho-1}$ gradually tends to 1. Consequently the point E closes to the point A as $\rho\to\infty$ and hence the line OA may be considered as the envelope for $\rho=\infty$. As well known, for a every bounded operator T the following inequality holds $1/2\|T\|\leqq\|T\|_N\leqq\|T\|$. Thus we may put

$$C_\infty=(\text{the set of all bounded operators})$$

and

$$\mathcal{D}_\infty=\text{the whole sector } \{(\|T\|, \|T\|_N)\ ;\ 1/2\|T\|\leqq\|T\|_N\leqq\|T\|\}.$$

When $\rho\to 2$, the slope $\frac{\rho-2}{2(\rho-1)}$ of $f_1(\rho)$ and the intercept $\frac{\rho}{2(\rho-1)}$ of $\|T\|_N$ gradually close to 0 and 1 respectively, that is, the points D and C gradually close to the same point B.

As stated in the previous paper [4] the triangular domain OAF and OAB indicate the necessary and sufficient conditions for T to belong to C_1 and C_2 respectively. The line OA indicates the degenerated domain which gives the necessary and sufficient condition for a normaloid* operator T to belong to C_ρ $(0\leqq\rho\leqq 1)$ ([4]).

*) An operator T is said to be *normaloid* if $\|T\|=\|T\|_N$ or equivalently $\|T\|$ equals to the spectral radius of T ([5]).

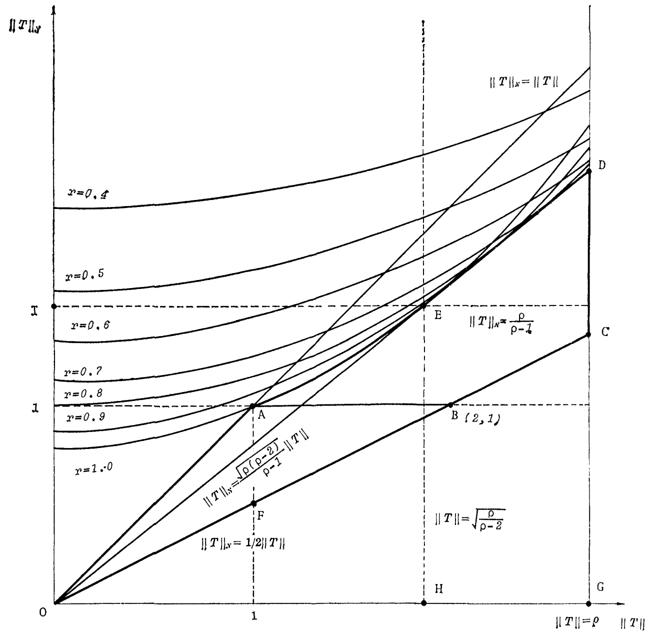

Fig. 1 $\rho \geqq \sqrt{2}+1$

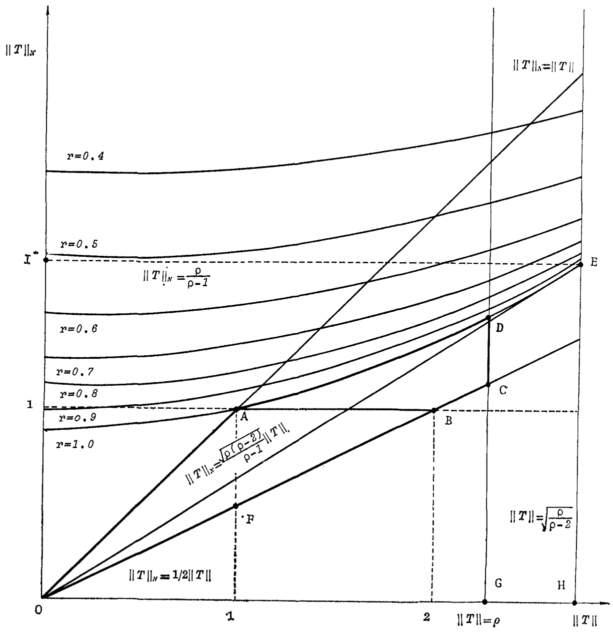

Fig. 2 $2 \leqq \rho \leqq \sqrt{2}+1$

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