79. *Generalizations of M-spaces. II*

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In the previous paper [4] we obtained a characterization of M'-spaces as a generalization of M-spaces and Morita's paracompactification of M'-spaces. In this paper we shall give necessary and sufficient conditions for an M'-space to be M-space and show that the product space of M'-spaces need not be an M'-space and that the property of being M'-space is not necessarily invariant under a perfect mapping (see [2] or [4] for terminologies and notations).

1. Relation between M'- and M-spaces.

A space X is a *cb-space* (resp. *weak cb-space*) if given a decreasing sequence $\{F_n\}$ of closed sets (resp. regular-closed sets) of X with empty intersection, there exists a sequence $\{Z_n\}$ of zero sets with empty intersection such that $F_n \subset Z_n$ for each n where a subset F is *regular-closed* if cl (int F)$=F$.

Lemma 1.1. *The following results has been obtained in* ([5], [6]).

1) *X is a cb-space if and only if X is both countably paracompact and week cb.*

2) *For a pseudocompact space X the followings are equivalent:* i) *X is a cb-space,* ii) *X is countably compact and* iii) *X is countably paracompact.*

3) *A countably compact space is a cb-space.*

4) *A pseudocompact space is a weak cb-space.*

The following lemma is obvious.

Lemma 1.2. *If $\{U_n\}$ is a decreasing sequence of open sets of X such that $\cap \bar{U}_n=\varnothing$, then*

1) *there exists a locally finite discrete collection $\{V_n\}$ of open sets of X such that $\bar{V}_n \subset U_n$ and $\bar{V}_n \cap \bar{V}_m=\varnothing$ $(n\neq m)$,*

2) *there exists a non-negative continuous function f on X such that $f=0$ on $X-\cup V_n$, $0\leq f\leq n$ on V_n and $f(x_n)=n$ for some point x_n of V_n, and*

3) *$\{Z_n; Z_n=\{x; f(x)\geq n\}\}$ is a decreasing sequence of zero sets of X with empty intersection.*

Theorem 1.3. *An M'-space is a weak cb-space.*

Proof. Let φ be an SZ-mapping from an M'-space X onto a metric space Y and $\{\mathfrak{V}_i; i\in N\}$ be a normal sequence of open covering of Y such that $\{\mathrm{St}(y, \mathfrak{V}_i); i\in N\}$ is a basis of neighborhoods at each point y

of Y. Let us put $\mathfrak{U}_i=\varphi^{-1}\mathfrak{B}_i$ $(i \in N)$. Then $\{\mathfrak{U}_i\,;\, i \in N\}$ satisfies the condition (M′) (cf. Theorem 6.1 in [7]). Now suppose that X is not weak cb, then there exists a decreasing sequence $\{F_i\}$ of regular-closed sets of X with empty intersection such that any sequence $\{Z_i\}$ of zero sets of X with $F_i \subset Z_i$ has a non-empty intersection. Since $\overline{\varphi(F_i)}$ is a zero set of Y, so is $\varphi^{-1}\overline{\varphi(F_i)}$. $F_i \subset \varphi^{-1}\overline{\varphi(F_i)}$ and there is a point x_0 such that $x_0 \in \cap \varphi^{-1}\overline{\varphi(F_i)}$ by the assumption. $y_0=\varphi(x_0) \in \overline{\varphi(F_i)}$ and $\mathrm{St}(y_0, \mathfrak{B}_i) \cap \varphi(F_i) \neq \emptyset$. This implies that $U_i=\mathrm{St}(x_0, \mathfrak{U}_i) \cap \mathrm{int}\, F_i \neq \emptyset$ because each F_i is regular-closed. Since $\cap F_i=\emptyset$, we have $\cap \bar{U}_i=\emptyset$. By Lemma 2 there exists a decreasing sequence $\{Z_i\}$ of zero sets such that $Z_i \subset \bigcup\{V_m\,;\, m\geq i\} \subset \mathrm{St}(x_0, \mathfrak{U}_i)$ and $\bigcap Z_i=\emptyset$. On the other hand $\{\mathfrak{U}_i\,;\, i \in N\}$ satisfies the condition (M′) and we have $\bigcap Z_i \neq \emptyset$. This is a contradiction, that is, X is a weak cb-space.

Lemma 1.4. *If X is countably paracompact and F is a relatively pseudocompact closed subset of X, then F is countably compact.*

Proof. Suppose that $\{x_n\,;\, n \in N\}$ is a sequence of points of F which has no accumulation points. $A_n=\{x_m\,;\, m\geq n\}$ is closed and $\bigcap A_n=\emptyset$. By the countable paracompactness there is a decreasing sequence $\{U_n\}$ of open sets such that $\bigcap \bar{U}_n=\emptyset$ and $x_n \in A_n \subset U_n$. Using (3) of Lemma 3.2 there exists a continuous function f on X such that $f(x_n)=n$ which contradicts the relatively pseudocompactness of F. Thus F must be countably compact.

Since an almost realcompact weak cb-space is realcompact (Theorem 1.2 in [1]), we have

Corollary 1.5. *If an M'-space is almost realcompact, then it is realcompact.*

From Theorem 1.4, Corollary 1.5 and Corollary 1.3 in [4], it is easy to see that the following theorem is a generalization of (2) of Lemma 1.1.

Theorem 1.6. *If X is an M'-space, then the followings are equivalent:*

1) *X is an M-space.*
2) *X is a cb-space.*
3) *X is countably paracompact.*

Proof. 2)↔3) follows from Theorem 1.3 and 1) of Lemma 1.1.

1)→2). Let φ be a quasi-perfect mapping from X onto a metric space Y and let $\{F_n\}$ be a decreasing sequence of closed sets of X with empty intersection. If $\bigcap\varphi(F_n)=\emptyset$, then there exists a sequence $\{Z'_n\}$ of zero sets of Y with $\bigcap Z'_n=\emptyset$. Thus $\{Z_n\,;\, Z_n=\varphi^{-1}(Z'_n)\}$ is a sequence of zero sets of X such that $\bigcap Z_n=\emptyset$. If $y_0 \in \bigcap\varphi(F_n)$, then $F_n \cap \varphi^{-1}(y_0) \neq \emptyset$ for each n. Since $\varphi^{-1}(y_0)$ is countably compact, and $\{F_n\}$ is decreasing, we have $\bigcap F_n \neq \emptyset$ which is impossible.

2)→1). Let φ be an SZ-mapping from X onto a metric space Y. By Lemma 1.4 it is sufficient to show that φ is closed. Let F be a closed subset of X and $y_0 \in \overline{\varphi(F)} - \varphi(F)$. Since Y is a metric space, there is a sequence $\{y_n\}$ which converges to y_0 and $y_n \in \varphi(F)$. $B_n = \{y_m ; m \geq n\} \cup \{y_0\}$ is a zero set of Y and $\{A_n = F \cap \varphi^{-1}(B_n)\}$ is a decreasing sequence of closed sets of X with $\bigcap A_n = \emptyset$. Since X is a cb-space, there exists a sequence $\{Z_n\}$ of zero sets of X such that $A_n \subset Z_n$ and $\bigcap Z_n = \emptyset$. φ being a Z-mapping, we have $y_0 \in \overline{\varphi(A_n)} \subset \varphi(Z_n)$. This shows that $\varphi_0^{-1}(y_0) \cap Z_n \neq \emptyset$. $\varphi^{-1}(y_0)$ being countably compact, we have $\bigcap Z_n \neq \emptyset$ which is a contradiction.

Corollary 1.7. *A pseudocompact M-space is countably compact.* This follows from Theorem 1.6 and 2) of Lemma 1.1.

2. Examples. The following example shows that there exists an M-space X such that some subspace W of μX, containing X, is not necessarily an M-space.

Example 2.1. Let A be a space $\{1/n ; n \in N\} \cup \{0\}$ with usual topology and ω_1 the first uncountable ordinal and $a_n = 1/2n$ ($n \in N$).

1) $X = A \times W(\omega_1)$ is countably compact [2] and hence an M-space.

2) $W = A \times W(\omega_1 + 1) - \{(a_n, \omega_1) ; n \in N\} - \{(0, \omega_1)\}$ is pseudocompact but not countably compact. Thus W is an M_δ-space but not an M-space by Corollary 1.7.

3) $X \subset W \subset \mu X = \upsilon X = \beta X$ is obvious.

Theorem 2.2. *If φ is an SZ-mapping from an M'-space X onto a topologically complete space Y, then Y is a paracompact M-space.*

Proof. As is known $\Phi^{-1}Y$ is topologically complete and $\mu X \subset \Phi^{-1}(Y)$ by Theorem 2.5 in [4]. Let us put $\varphi_1 = \Phi | X$. Similarly to the proof of Theorem 2.5 in [4], φ_1 becomes a perfect mapping from μX onto Y. Thus Y must be a paracompact M-space by Lemma 2.3 in [4].

In Theorem 2.2 we can not drop the topological completeness of Y. Such an example is given in the following and it is an example showing that an image of M'-space under a perfect mapping need not be an M'space

Example 2.3. There exists a locally compact, non-normal, countably paracompact nonweak cb-space Y which is an image of an M-space under a perfect mapping (and hence Y is not an M'-space).

The example given here is a space constructed by K. Morita ([8], § 4) (an analogous example was given in § 3 in [6]). Let $S = W(\omega_1 + 1)$ $W(\omega_1 + 1) - (\omega_1, \omega_1)$, $P = \{(\alpha, \omega_1) ; \alpha < \omega_1\}$ and $Q = \{(\omega_1, \beta) ; \beta < \omega_1\}$. Let X be the topological sum of disjoint spaces S_n where for each $n \in N$, there is a homeomorphism φ_n of S onto S_n. Then X is non-normal, locally compact, countably paracompact M-space. Now we identify a point $\varphi_{2m-1}(p)$ with $\varphi_{2m}(p)$ for $p \in P$ and a point $\varphi_{2m}(q)$ with $\varphi_{2m+1}(q)$ for $q \in Q$. By this identification, we have an identification space Y and the identification mapping $\varphi; X \to Y$. It is obvious that φ is perfect. Thus Y is locally compact, non-normal and countably paracompact. If Y is an M'-space, then by Theorem 1.6 Y must be an M-space. But it is shown by K. Morita that Y is not an M-space. Thus Y is not an M'-space. To show that Y is not a weak cb-space we put $F_n = \mathrm{cl}\left(Y - \varphi\left(\bigcup_{i=1}^{n} \varphi_i(S_i)\right)\right)$. Then $\{F_n\}$ is a decreasing sequence of regular closed-sets of Y. Similarly to Morita's example [8] it is proved that there are no sequence $\{Z_n\}$ of zero sets of Y such that $F_n \subset Z_n$ for each $n \in N$ and $\bigcap Z_n = \varnothing$.

The following example shows that a product of M'-spaces need not be an M'-space.

Example 2.4. In [3], we proved the following theorem: Suppose that X is not pseudocompact and P and Q are disjoint non-empty subset of $\beta X - X$. If $X \cup P$ and $X \cup Q$ are countably compact, then $A \times B$ is not an M-space where $A = X \cup P \cup \{x^*\}$, $B = X \cup Q \cup \{x^*\}$ and x^* is an arbitrary point contained in $\beta X - \upsilon X$. If $X = N$ and we take both subsets P and Q such that $\beta N - N = P \cup Q$, $P \cap Q = \varnothing$ and both subspace $N \cup P$ and $N \cup Q$ are countably compact as in [9] (or, see [3]) then the set K_n constructed in the proof of Theorem 1 in [3] is open-closed and hence it is a zero set. Since the sequence $\{K_n\}$ has a empty total intersection, this shows that the condition (M′) does not hold and hence $A \times B$ is not an M'-space.

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