93. *A Remark on a Conjecture of Paley*

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The standard symbols of the Nevanlinna theory

$$\log^+, M(r, f), m(r, a), N(r, a), T(r, f), \delta(a, f)$$

are used throughout this note. We define

$$N(r) = N(r, 0) + N(r, \infty)$$

and

$$K(f) = \limsup_{r\to\infty} \frac{N(r)}{T(r)}.$$

Paley [3] conjectured that an integral function of finite order $\rho > \frac{1}{2}$ satisfies

$$\limsup_{r\to\infty} \frac{m(r, f)}{\log M(r, f)} \geqq \frac{1}{\pi\rho}.$$

The object of the present note is to show that as an *immediate consequence* of Edrei-Fuchs's results [1, 2] we obtain

Theorem. *If an integral function of finite order* $\rho > \frac{1}{2}$ *satisfies*

$$\sum_{a\neq\infty} \delta(a, f) = 1, \tag{1}$$

then we have

$$\frac{1}{2} \geqq \limsup_{r\to\infty} \frac{m(r, f)}{\log M(r, f)} \geqq \liminf_{r\to\infty} \frac{m(r, f)}{\log M(r, f)} \geqq \frac{1}{\pi}.$$

In particular if there exists a finite a with $\delta(a, f) = 1$, *then*

$$\lim_{r\to\infty} \frac{m(r, f)}{\log M(r, f)} = \frac{1}{\pi}. \tag{2}$$

Edrei and Fuchs proved the following theorem and lemmas.

Theorem A [1]. *If the integral function* $f(z)$ *in question satisfies* (1), *then*

$$\lim_{r\to\infty} \frac{T(r, f')}{T(r, f)} = 1, \quad K(f') = 0,$$

and $f(z)$ *is necessarily of positive integral order and of regular growth.*

Lemmas [2]. *Let* $f(z)$ *be a meromorphic function of finite lower order* μ *and* p *be the non-negative integer defined by the inequalities*

$$p - \frac{1}{2} \leqq \mu < p + \frac{1}{2}.$$

Let $E(u, p)$ *be the primary factor of genus* p. *Now suppose that the function* $f(z)$ *satisfies*

$$K(f)<\frac{\varepsilon}{B_0(p+1)\log(p+1)+B_1(p+1)\log(1/\delta)},$$

where $0<\varepsilon\leqq 1$, $0<\delta<e^{-1}$, $B_1\leqq B_0$ *and* B_1 *is a sufficiently large number.* Then we obtain the following I, II and III:

Lemma I. $p\geqq 1$ *and* $f(z)$ *has the representation*

$$f(z)=z^k e^{\alpha_0 z^p+\alpha_1 z^{p-1}+\cdots+\alpha_p}\frac{E\left(\frac{z}{a_\nu},p\right)}{E\left(\frac{z}{b_\nu},p\right)} \quad (k \text{ integer}).$$

Lemma II. *We set* $\alpha=e^{1/(p+1)}$ *and*

$$c_j=\alpha_0+\frac{1}{p}\left\{\sum_{|a_\nu|\leqq\alpha^j} a_\nu^{-p}-\sum_{|b_\nu|\leqq\alpha^j} b_\nu^{-p}\right\}.$$

Consider the annulus Γ_j defined by

$$\alpha^j\leqq r<\alpha^{j+\frac{2}{3}} \qquad (j=1,2,\cdots;\ z=re^{i\theta}).$$

Then we have

$$T(r)<\frac{4}{\pi}|c_j|r^p,\quad \alpha^j\leqq r<\alpha^{j+\frac{2}{3}},\quad j\geqq j_0.$$

Lemma III. *For all sufficiently large integer j we may find an exceptional set E_j, such that*

$$z\in\{\Gamma_j-E_j\}$$

implies

$$|\log|f(z)|-Rc_j z^p|<4\varepsilon|c_j|r^p,$$

and E_j is covered by circles subtending angles at the origin whose sum S_j does not exceed $8\pi e^3\delta$. In particular if $f(z)$ is an entire function, we have

$$\log|f(z)|<Rc_j z^p+4\varepsilon|c_j|r^p$$

for $z\in\Gamma_j(j>j_0)$.

Proof of Theorem. By Theorem A we have $K(f')=0$, and thus we apply lemmas to $f'(z)$. Let $\eta>0$ be a sufficiently small number and set

$$\delta=\frac{\eta}{4\pi e^3 p},\tag{3}$$

then by Lemma III we obtain

$$\log M(r,f')>|c_j|r^p\cos\frac{\eta}{p}-4\varepsilon c_j r^p$$

for $\alpha^j\leqq|z|=r<\alpha^{j+\frac{2}{3}}(j\geqq j_0)$, and

$$\begin{aligned} m(r,f')&=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f'(re^{i\theta})|d\theta\\ &\leqq\frac{1}{2\pi}\cdot p\cdot\int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}}|c_j|r^p\cos p\theta d\theta+4\varepsilon|c_j|r^p\\ &=\frac{1}{\pi}|c_j|r^p+4\varepsilon|c_j|r^p. \end{aligned}$$

Hence

$$\frac{m(r,f')}{\log M(r,f')} \leqq \frac{\frac{1}{\pi}(1+4\varepsilon\pi)}{\cos\frac{\eta}{p}-4\varepsilon}. \tag{4}$$

Moreover the condition (3) implies that S_j equals at most $2\eta/p$. This gives

$$\begin{aligned} m(r, f') &\geqq \frac{1}{2\pi}\cdot p\cdot\int_{-\frac{\pi}{2p}}^{-\frac{\eta}{p}} |c_j| r^p \cos p\theta\, d\theta + \frac{1}{2\pi}\int_{\frac{\pi}{p}}^{\frac{\eta}{2p}} |c_j| r^p \cos p\theta d\theta - 4\varepsilon |c_j| r^p \\ &= \frac{1}{\pi}|c_j| r^p (1-\sin\eta - 4\varepsilon\pi) \qquad (j\geqq j_0). \end{aligned} \tag{5}$$

Therefore we have

$$\frac{m(r,f')}{\log M(r,f')} \geqq \frac{\frac{1}{\pi}(1-\sin\eta-4\varepsilon\pi)}{1+4\varepsilon} \qquad (r\geqq r_0).$$

Since $\varepsilon>0$ and $\eta>0$ may be chosen as small as possible, from (4) and (5) we deduce

$$\frac{1}{\pi}(1+0(1)) \geqq \frac{m(r,f')}{\log M(r,f')} \geqq \frac{1}{\pi}(1-0(1)) \qquad (r\to\infty). \tag{6}$$

This and Theorem A give

$$\liminf_{r\to\infty} \frac{m(r,f)}{\log M(r,f)} \geqq \frac{1}{\pi}$$

with the aid of the well known inequality

$$\log M(r,f) \leqq \log M(r,f') + 0(\log r).$$

Next we shall prove

$$\limsup_{r\to\infty} \frac{m(r,f)}{\log M(r,f)} \leqq \frac{1}{2}. \tag{7}$$

Let $\arg c_j = \omega_j$. We denote by A_j and B_j the sets of points $z=re^{i\theta}$, which belong to Γ_j, defined by $\cos(p\theta+\omega_j) \geqq -5\varepsilon$ and $\cos(p\theta+\omega_j) < -5\varepsilon$ respectively. B_j consists of p circular rectangles which we denote by $B_j^{(1)}, B_j^{(2)}, \cdots, B_j^{(p)}$. Edrei-Fuchs proved that every $B_j^{(i)}(i=1,2,\cdots,p)$ meets necessarily one (which we denote by $\mathcal{L}^{(i)}$) of the asymptotic paths $\mathcal{L}^{(1)}, \mathcal{L}^{(3)}, \cdots$ possessing finite asymptotic values [2]. By Lemmas II and III for all sufficiently large r we deduce

$$|f'(z)| < e^{-\frac{\pi\varepsilon}{4}T(r,f')},\ z\in B_j.$$

Now for arbitrary z belonging to $B_j^{(i)}$ and z_{ij} on $\mathcal{L}^{(i)} \cap B_j^{(i)}$ we have

$$|f(z)-f(z_{ij})| = \left|\int_{z_{ij}}^{z} f'(z)dz\right| \leqq K\cdot re^{-\frac{\pi\varepsilon}{4}T(r,f')},\ (j\geqq j_0),$$

where $K>0$ is an absolute constant. Since the right-hand side converges to zero as $r\to\infty$ and $\lim f(z_{ij})=\beta_i$ is finite, $f(z)$ converges to β_i uniformly in $B_j^{(i)}$ as $j\to\infty$. Therefore for all sufficiently large r

$$m(r, f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$$
$$\leqq \frac{1}{2\pi}\log M(r, f)\cdot\frac{2\ \mathrm{Arc}\cos(-5\varepsilon)}{p}\cdot p+0(1)$$

and thus

$$\frac{m(r, f)}{\log M(r, f)}\leqq\frac{1}{\pi}\ \mathrm{Arc}\cos(-5\varepsilon)+0(1) \qquad (r\to\infty).$$

As $\varepsilon>0$ may be chosen as small as we please, this gives (7). We shall prove the latter of theorem. If we set $F(z)=f(z)-a$, then $\delta(0, F)=\delta(a, f)=1$, and $K(F)=0$. Therefore we have (6) with $F(z)$ instead of $f'(z)$, and (2).

References

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