127. *Surjectivity of Linear Mappings and Relations*

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(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1969)

In [3], Pták has proved the following theorem, in which (1) is called the closed relation theorem and (2) the open mapping theorem.

Theorem A. *Let E be a Banach space, F a normed linear space, R a closed linear subspace of $E\times F$, T a continuous linear mapping of E into F, and let $0<\alpha<\beta$. Let U and V be the unit balls of E and F respectively.*

(1) *If the set $RU+\alpha V$ contains a translate of βV, then $RE=F$ and $(\beta-\alpha)V\subset RU$.*

(2) *If the set $T(U)+\alpha V$ contains a translate of βV, then $T(E)=F$ and $(\beta-\alpha)V\subset T(U)$, so that T is open.*

A theorem which is similar to the assertion (2) is obtained by McCord [2]:

Theorem B. *Suppose T is a continuous linear mapping of a Banach space E into a normed linear space F, for which there are positive real numbers α and β, $\beta<1$, such that the following holds. For each y in F of norm 1, there exists an x in E of norm $\leq\alpha$ such that $\|y-Tx\|\leq\beta$. Then for each y in F, there exists an x in E such that $y=Tx$ and $\|x\|<\alpha(1-\beta)^{-1}\|y\|$.*

Theorem A has been generalized by Baker [1]. In this paper we shall state other generalizations of Theorem A and a generalization of Theorem B.

Throughout this paper, vector spaces are over the real or the complex numbers. Let E and F be two vector spaces, A a subset of E, and R be a subset of $E\times F$. By $R(A)$ we denote the set of all $y\in F$ such that $(x, y)\in R$ for some $x\in A$; the set $R(\{x\})$, where $x\in E$, will be denoted by $R(x)$. $S(A)$ denotes the union of all λA with λ in the closed unit interval $[0, 1]$, and A is said to be *star-shaped* if $S(A)=A$.

The essential part of our results is included in the following

Lemma. *Let E and F be two topological vector spaces, and R be a closed vector subspace of $E\times F$. Let B_0 be a sequentially complete bounded star-shaped convex subset of E such that $R(B_0)\neq\varnothing$, and let B be a bounded subset of F. Then $B\subset R(B_0)+\alpha B$ implies $(1-\alpha)B\subset R(B_0)$ for every $\alpha\in[0, 1)=[0, 1]\backslash\{1\}$.*

Proof. It suffices to consider the case where $\alpha\neq 0$. Let y be an arbitrary element of B. Since $B\subset R(B_0)+\alpha B$, there are points $x_1\in B_0$

and $y_1 \in R(x_1)$ such that $y-y_1 \in \alpha B \subset R(\alpha B_0)+\alpha^2 B$. Therefore we have, for some $x_2 \in \alpha B_0$ and for some $y_2 \in R(x_2)$, $y-y_1-y_2 \in \alpha^2 B \subset R(\alpha^2 B_0) + \alpha^3 B$. Thus we can find recursively two sequences $\{x_n \mid n=1, 2, \cdots\}$ and $\{y_n \mid n=1, 2, \cdots\}$ satisfying the following conditions 1)–3) for every $n=1, 2, \cdots$.

1) $$x_n \in \alpha^{n-1} B_0.$$

2) $$y_n \in R(x_n).$$

3) $$y-\sum_{i=1}^{n} y_i \in \alpha^n B.$$

Since B_0 is star-shaped and convex, we have

$$\sum_{i=1}^{n} x_i \in B_0+\alpha B_0+ \cdots +\alpha^{n-1} B_0 \subset \frac{1-\alpha^n}{1-\alpha} B_0 \subset \frac{1}{1-\alpha} B_0$$

for every $n=1, 2, \cdots$. From the boundedness of B_0 and 1), it is easy to see that the sequence $\left\{\sum_{i=1}^{n} x_i \mid n=1, 2, \cdots\right\}$ is a Cauchy sequence. Consequently, the sequence $\left\{\sum_{i=1}^{n} x_i \mid n=1, 2, \cdots\right\}$ converges to an element $x \in \dfrac{1}{1-\alpha} B_0$, since B_0 is sequentially complete. Now the condition 2) shows $(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i) \in R$, and the condition 3) implies, because of the boundedness of B, that the sequence $\left\{\sum_{i=1}^{n} y_i \mid n=1, 2, \cdots\right\}$ converges to y. Thus we have $(x, y) \in R$ or $(1-\alpha)y \in R(B_0)$, which establishes the lemma.

If $y \in R(x)$, then we have $B \subset R(B_0)+\alpha B$ for $B_0=S(x)$, $B=\{y\}$, and $\alpha=0$. Therefore we have the following

Theorem 1. *Let E and F be two topological vector spaces, R a closed vector subspace of $E \times F$, and let $y \in F$. If $y \in R(x)$ for some $x \in E$, then*

(*) *there exist a sequentially complete bounded star-shaped convex subset B_0 of E and a bounded subset $B \subset F$ containing y such that $R(B_0) \neq \varnothing$ and $B \subset R(B_0)+\alpha B$ for some $\alpha \in [0, 1)$.*

Conversely, if the condition (*) is satisfied, then $y \in R(x)$ for some

$$x \in \frac{1}{1-\alpha} B_0.$$

The graph of a continuous linear mapping of a topological vector space E into a Hausdorff topological vector space F is a closed vector subspace of $E \times F$. Hence the following corollary is evident.

Corollary 1. *Let u be a continuous linear mapping of a topological vector space E into a Hausdorff topological vector space F, and let $y \in F$. If $y=u(x)$ for some $x \in E$, then*

(**) *there exist a sequentially complete bounded star-shaped con-*

vex subset B_0 of E and a bounded subset $B\subset F$ containing y such that $B\subset u(B_0)+\alpha B$ for some $\alpha\in[0, 1)$.

Conversely, if the condition $(**)$ is satisfied, then $y=u(x)$ for some

$$x\in\frac{1}{1-\alpha}B_0.$$

Theorem 1 yields obviously the following

Theorem 2. *Under the hypothesis of Theorem 1, $R(E)=F$ if and only if there exists a family $\mathscr{B}$ of bounded subsets of F such that the union of all members of $\mathscr{B}$ spans F and there correspond, to each $B\in\mathscr{B}$, a sequentially complete bounded star-shaped convex subset B_0 of E and an $\alpha\in[0, 1)$ satisfying the relations: $R(B_0)\neq\emptyset$ and $B\subset R(B_0)+\alpha B$.*

Corollary 2. *Under the hypothesis of Corollary 1, the mapping u is surjective if and only if there exists a family $\mathscr{B}$ of bounded subsets of F such that the union of all members of $\mathscr{B}$ spans F and there correspond, to each $B\in\mathscr{B}$, a sequentially complete bounded star-shaped convex subset B_0 of E and an $\alpha\in[0, 1)$ satisfying the relation $B\subset R(B_0)+\alpha B$.*

Another consequence of Theorem 1 is the following

Theorem 3. *Under the hypothesis of Theorem 1, if there exist a sequentially complete bounded star-shaped convex subset B_0 of E, a bounded subset B of F, a subset A of F absorbing each non-zero element of F, and an $\alpha\in[0, 1)$ such that $R(B_0)\neq\emptyset$, $B\subset S(A)$ and $A\subset R(B_0)+\alpha B$, then $(1-\alpha)A\subset R(B_0)$, and hence $R(E)=F$.*

In fact, since $R(B_0)$ is star-shaped convex, we have $S(B)\subset S(A)\subset R(B_0)+\alpha S(B)$, and so $(1-\alpha)S(B)\subset R(B_0)$; consequently we have

$$(1-\alpha)A\subset(1-\alpha)S(A)\subset(1-\alpha)R(B_0)+\alpha(1-\alpha)S(B)$$
$$\subset(1-\alpha)R(B_0)+\alpha R(B_0)\subset R(B_0).$$

The following corollary is a generalization of Theorem B.

Corollary 3. *Under the hypothesis of Corollary 1, if there exist a sequentially complete bounded star-shaped convex subset B_0 of E, a bounded subset B of F, a subset A of F absorbing each non-zero element of F, and an $\alpha\in[0, 1)$ such that $B\subset S(A)$ and $A\subset u(B_0)+aB$, then $(1-\alpha)A\subset u(B_0)$, and so u is surjective.*

Remark. The well-known fact "a Hausdorff topological vector space E having a precompact neighborhood of 0 is of finite dimensional" follows immediately from the above lemma. In fact, since a precompact neighborhood is bounded, it is sufficient to show that if a bounded set B of E is covered by a finite number of translations of αB for some $\alpha\in[0, 1)$, then B spans a finite dimensional vector subspace of E. Now let $B\subset\bigcup_{i=1}^{n}(a_i+\alpha B)$, $a_1, a_2, \cdots, a_n\in E$. Then $B\subset e(S(A))+\alpha B$, where A is the convex hull of the set $\{a_1, \cdots, a_n\}$ and e is the identity mapping of E into itself. Since $S(A)$ is sequentially complete and bounded, by the above lemma we have $(1-\alpha)B\subset S(A)$ from which the desired conclusion follows.

References

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