1. *On Functions of Yosida's Class* (A)

By V. I. GAVRILOV*)
Department of Mathematics, University of Moscow, U.S.S.R.

(Comm. by Zyoiti SUETUNA, M. J. A., Jan. 12, 1970)

1. Let $f(z)$ be a non-rational meromorphic function in $|z|<\infty$ and $\rho(f(z))$ the spherical derivative of $f(z)$. Following K. Yosida [2], we say that $f(z)$ belongs to the class (A) if for any sequence of complex numbers $\{a_n\}$, the family of functions

$$\{f(z+a_n)\}, \qquad n=1,2,\cdots, \tag{1}$$

is normal in the sense of Montel $|z|<\infty$. If, in addition, any family of the form (1) admits no constant limit, we say that $f(z)$ belongs to the subclass (A_0) [the functions of the 1st category in Yosida's terminology]. The subclass (A_0) contains, in particular, an important class of meromorphic functions as the doubly periodic functions.

Yosida [2] has proved that $f(z)$ belongs to (A) if and only if

$$\rho(f(z))=O(1), \qquad z\to\infty.$$

Among the other results, he has proved that a function of the subclass (A_0) possesses no Nevanlinna deficient value. In [1] the author has pointed out that Yosida's results allow to prove that a function of (A_0) admits no Valiron deficient value. The present note contains the details.

2. Using the standard terminology of the Nevanlinna theory, the deficiency of Valiron $\delta(a,f)$ of a value a is defined as follows:

$$\delta(a,f)=\overline{\lim_{r\to\infty}}\frac{m(r,a,f)}{T(r,f)}.$$

If $\delta(a,f)>0$, the value a is said to be a Valiron deficient value for $f(z)$.

Theorem. *If $f(z)$ belongs to (A_0), then $\delta(a,f)=0$ for any complex a (finite or infinite).*

Proof. Yosida [2] has proved that for a function $f(z)\in(A_0)$ and for a set of complex values $a_1, a_2, \cdots, a_q$ $(q\geq 3)$,

$$\sum_{i=1}^{q} m(r,a_i,f)=O(r)+S(r)$$

holds with $S(r)=o(T(r,f))$. Our theorem will be proved if we show that

$$\lim_{r\to\infty}\frac{T(r,f)}{r^2}>0 \tag{2}$$

is valid for any $f(z)\in(A_0)$.

*) At present Visiting Professor, Department of Mathematics, Indian Institute of Technology, Bombay (India).

To prove (2) we make use of the Shimizu-Ahlfors formula

$$T(r, f)+O(1)=\int_0^r \frac{dt}{t}\iint_{|z|<t} [\rho(f(z))]^2 dxdy,\ z=x+iy, \qquad (3)$$

and a theorem of Yosida [2]: $f(z)$ belongs to (A_0) if and only if for any $\delta>0$ there exists an $\varepsilon=\varepsilon(\delta)>0$ such that

$$\iint_{|z-z_0|<\delta} [\rho(f(z))]^2 dxdy \geq \varepsilon \qquad (4)$$

holds for any disk $|z-z_0|<\delta$ of radius δ in $|z|<\infty$.

We put $\delta=\frac{1}{2}$ and denote the corresponding value of $\varepsilon=\varepsilon(\delta)$ by $\varepsilon_0>0$. Consider a disk $|z|<t$, $t>2$, and divide it into the annuli Δ_k: $k-1\leq|z|<k$; $k=1, 2, \cdots, [t]$; here $[t]$ denotes the integral part of t. For a fixed k, the annulus Δ_k contains at least $2k-1$ mutually disjoint disks of radius $\frac{1}{2}$. Thus, the number of mutually disjoint disks of radius $\frac{1}{2}$, which are contained in $|z|<t$, $t>2$, is greater than $[t]^2>\frac{t^2}{4}$. Therefore, by (4), for any $f(z)\in(A_0)$ the right hand side of (3) is of the form $\frac{\varepsilon_0 r^2}{8}+O(1)$, which proves (2).

Remark. From the point of view of the classification of meromorphic functions given in [1], it is interesting to compare our theorem to a result of T. Zinno and N. Toda [3]: if $f(z)$ satisfies

$$\rho(f(z))=O\left(\frac{1}{|z|}\right), \qquad z\to\infty,$$

then it admits no Valiron deficient value.

References

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[3] T. Zinno and N. Toda: On Julia's exceptional functions. Proc. Japan Acad., **42** (10), 1120–1121 (1968).