35. *Notes on Semilattices of Groups*

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Recently a lot of ideal theoretical characterizations for semigroups which are semilattices of groups were given by the author (see [2], [3]). Continuing these investigations several further criteria will be established here. For the terminology we refer to A. H. Clifford and G. B. Preston's books [1] and for the definition of (m, n)-ideals see the author's paper [5].

Theorem 1. *An arbitrary semigroup S is a semilattice of groups if and only if the relation*

(1) $$L \cap B = LB$$

holds for any bi-ideal B and for any left ideal L of S.

Proof. *Necessity.* Let S be a semigroup which is a semilattice of groups. Then it is regular and every one-sided ideal of S is two-sided (see Exercise 4.2.2 in [1], I). In this case every bi-ideal B of S is also a two-sided ideal of S by a recent result of the author [4]. Therefore (1) follows from the well known regularity criterion:

(2) $$L \cap R = RL$$

for any left ideal L and for any right ideal R of S.

Sufficiency. Let S be a semigroup with property (1) for any left ideal L and for any bi-ideal B of S. Then (1) implies

(3) $$S \cap R = SR$$

for any right ideal R of S, and

(4) $$L \cap S = LS$$

for any left ideal L of S, that is, every one-sided ideal of S is two-sided. Thus we obtain that $A \cap B = AB$ for any two two-sided ideals A, B of S, i.e. S is regular. Next we show that S is a centric semigroup. Indeed, for any element a of S the equality (1) implies

(5) $$aS = S \cap aS = SaS,$$

and

(6) $$Sa = Sa \cap S = SaS.$$

(5) and (6) imply that $aS = Sa$ for any element a in S. It is known[1] that the idempotent elements of a centric semigroup lie in the center, thus $ef = fe$ for any two idempotent elements of S. Therefore S is an inverse semigroup every one-sided ideal of which is two-sided. This means that S is a semigroup which is a semilattice of groups.

1) See Clifford and Preston [1], II.

The following criteria can be proved analogously.

Theorem 2. *A semigroup S is a semilattice of groups if and only if any one of the following conditions holds:*

(i) $L \cap Q = LQ$ *for any left ideal* L *and for any quasi-ideal* Q *of* S.

(ii) $Q \cap R = QR$ *for any right ideal* R *and for any quasi-ideal* Q *of* S.

(iii) $B \cap R = BR$ *for any bi-ideal* B *and for any right ideal* R *of* S.

More generally we have the result as follows.

Theorem 3. *For a semigroup S the conditions* (A)-(C) *are equivalent:*

(A) *S is a semilattice of groups.*

(B) $A \cap B = AB$ *for every* (m, m)*-ideal* A *and for every* $(n, 0)$*-ideal* B *of* S.

(C) $A \cap B = AB$ *for any* $(0, n)$*-ideal* A *and for any* (m, m)*-ideal* B *of* S (*m and n being arbitrary fixed positive integers*).

References

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