63. *Asymptotic Property of Solutions of Some Higher Order Hyperbolic Equations.* II

By Kiyoshi MOCHIZUKI
Yoshida College, Kyoto University

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1970)

3. In this part, we consider the inhomogeneous equation

$$(2)' \qquad \prod_{j=1}^{m}[\partial_t^2+\alpha_j L]u(t)=ge^{i\omega t},$$

where $g\in X$ and $\omega\neq 0$ real. We restrict ourselves to the case when the Hilbert space X and the operator $H=L^{1/2}$ satisfy the following conditions, and prove the so called limiting amplitude principle.

[C.1] There exists a Fréchet space Y, into which X is densely injected, with semi-norms $\{\rho_\nu(f)=[\rho_\nu(f,f)]^{1/2};\ \nu=1,2,\cdots\}$ having the following properties:

$$(28) \qquad \rho_\nu(f)\leq\rho_{\nu+1}(f)\leq\|f\| \text{ and } \sup_\nu \rho_\nu(f)=\|f\| \text{ for all } f\in X.$$

[C.2] The set X' defined below is dense in X.

Definition. We denote by X' the set of all $g\in X$ which satisfy the following two conditions:

(i) Let $[a,b]$ be any bounded interval in $\boldsymbol{R}^1_+$. Then, as $\varepsilon\to\pm 0$, $(H-\sigma-i\varepsilon)^{-1}g$ converges uniformly in $\sigma\in[a,b]$ in the sense of each ρ_ν-topology.

(ii) We put $(H-\sigma\mp i0)^{-1}g\equiv\lim_{\varepsilon\to\pm 0}(H-\sigma-i\varepsilon)^{-1}g$. Then $(H-\sigma\mp i0)^{-1}g$ is a Hölder continuous function of $\sigma\in\boldsymbol{R}^1_+$ with values in Y.

[C.3] The origin 0 is not an eigenvalue of H.

Now, by the same reasoning as in the proof of Theorem 3, we see that the initial value problem (2)′, (3) has a unique solution in the class $\bigcap_{0\leq j\leq 2m}\mathcal{E}_t^j(D(H^{2m-j+1}))$. Further, it follows that

$$(29) \qquad H^{2m-j}\partial_t^{j-1}u(t)=\sum_{k=1}^{2m}(\gamma_k)^{j-1}e^{\gamma_k Ht}\sum_{l=1}^{2m}n_{kl}H^{2m-l}\varphi_l + \sum_{k=1}^{2m}(\gamma_k)^{j-1}\int_0^t e^{\gamma_k H(t-s)}n_{k2m}ge^{i\omega s}ds$$

(cf., (26)).

Lemma 3. *If we choose $g\in X'$, then as $t\to\infty$*

$$(30) \qquad H^{2m-j}\partial_t^{j-1}u(t)\to ie^{i\omega t}\sum_{k=1}^{2m}(\gamma_k)^{j-1}(-i\gamma_k H-\omega+i0)^{-1}n_{k2m}g$$

in the sense of each ρ_ν-topology.

Proof. Note that for any $\gamma\neq 0$ pure imaginary and $f\in X$,

$$e^{\gamma Ht}f=\int_0^\infty e^{\gamma\sigma t}dE_\sigma^H f$$

and
$$\int_0^t e^{\gamma H(t-s)} f e^{i\omega s} ds$$
$$= s\text{-}\lim_{\varepsilon\to+0}\int_0^\infty e^{(\gamma\sigma-\varepsilon)t}\left[\int_0^t e^{(-\gamma\sigma+i\omega+\varepsilon)s}ds\right]dE_\sigma^H f^{3)}$$
$$= s\text{-}\lim_{\varepsilon\to+0}\int_0^\infty \frac{e^{i\omega t}-e^{(\gamma\sigma-\varepsilon)t}}{-\gamma\sigma+i\omega+\varepsilon}dE_\sigma^H f.$$

Then it follows from (29) that

$$H^{2m-j}\partial_t^{j-1}u(t)=\sum_{k=1}^{2m}(\gamma_k)^{j-1}\int_0^\infty e^{\gamma_k\sigma t}dE_\sigma^H\tilde{\varphi}_k + \sum_{k=1}^{2m}(\gamma_k)^{j-1}\, s\text{-}\lim_{\varepsilon\to+0}\int_0^\infty \frac{e^{i\omega t}-e^{(\gamma_k\sigma-\varepsilon)t}}{-\gamma_k\sigma+i\omega+\varepsilon}dE_\sigma^H\tilde{g}_k, \tag{31}$$

where we put $\tilde{\varphi}_k=\sum_{l=1}^{2m}n_{kl}H^{2m-l}\varphi_l$ and $\tilde{g}_k=n_{k2m}g$.

Given any $\varepsilon>0$, there exists $\tilde{\psi}_k\in X'$ such that
$$\|\tilde{\varphi}_k-\tilde{\psi}_k\|<\varepsilon$$
by [C.2]. On the other hand, if we note [C.3], then there exists a sufficiently large $r=r(\varepsilon)$ such that
$$\|\tilde{\psi}_k-\{E_r^H-E_{1/r}^H\}\tilde{\psi}_k\|<\varepsilon.$$
It follows from (i) in Definition that
$$dE_\sigma^H\tilde{\psi}_k=\{(H-\sigma-i0)^{-1}\tilde{\psi}_k-(H-\sigma+i0)^{-1}\tilde{\psi}_k\}d\sigma/2\pi i.$$
Hence, by the Riemann-Lebesgue theorem we have
$$\rho_\nu\left(\int_{1/r}^r e^{\gamma_k\sigma t}dE_\sigma^H\tilde{\psi}_k\right)\to 0 \quad \text{as } t\to\infty.$$
Summing up we conclude that as $t\to\infty$, the first term of the right member of (31) tends to zero in the sense of each ρ_ν-topology.

For the second term, we have
$$s\text{-}\lim_{\varepsilon\to+0}\int_0^\infty \frac{e^{i\omega t}-e^{(\gamma_k\sigma-\varepsilon)t}}{-\gamma_k\sigma+i\omega+\varepsilon}dE_\sigma^H\tilde{g}_k=ie^{i\omega t}(-i\gamma_k H-\omega+i0)^{-1}\tilde{g}_k + \lim_{\varepsilon\to+0}\int_0^\infty \frac{e^{(\gamma_k\sigma-\varepsilon)t}}{\gamma_k\sigma-i\omega-\varepsilon}dE_\sigma^H\tilde{g}_k,$$
where the last limit is taken in the sense of ρ_ν-topology. It is not difficult to see that
$$\rho_\nu\left(\lim_{\varepsilon\to+0}\int_0^\infty \frac{e^{(\gamma_k\sigma-\varepsilon)t}}{\gamma_k\sigma-i\omega-\varepsilon}dE_\sigma^H\tilde{g}_k\right)\to 0 \quad \text{as } t\to\infty$$
if we note (ii) in Difinition and
$$\lim_{t\to\infty}\lim_{\varepsilon\to+0}\int_e \frac{e^{(i\sigma-\varepsilon)t}}{\sigma+i\varepsilon}d\sigma=0$$
for each interval e which includes the origin. q.e.d.

Lemma 4. *For any $f\in X$ and $\kappa=i\omega+\varepsilon$ $(\varepsilon\neq 0)$, the reduced equation*

$$\prod_{j=1}^{m}[\kappa^2+\alpha_j L]v_\kappa\equiv\prod_{j=1}^{2m}[\kappa+\gamma_j H]v_\kappa=f \tag{32}$$

3) s-lim means the strong limit in X.

has a unique solution $v_\kappa \in \mathcal{D}(H^{2m})$. *Further, it follows that*

(33) $$H^{2m-j}\kappa^{j-1}v_\kappa = -i\sum_{k=1}^{2m}(\gamma_k)^{j-1}(-i\gamma_k H + i\kappa)^{-1}n_{k2m}f.$$

Proof. We put $V_\kappa = {}^t(v_\kappa, \kappa v_\kappa, \cdots, \kappa^{2m-1}v_\kappa)$ and $F = {}^t(0, 0, \cdots, f)$. Then it follows from (32) that

$$(\kappa - iDH)NE(H)V_\kappa = NF.$$

Since $(\kappa - iDH)$ is invertible, we have

$$E(H)V_\kappa = N^{-1}(\kappa - iDH)^{-1}NF,$$

which proves (33). q.e.d.

From (30) and (33), we now get the following theorem which asserts the limiting amplitude principle.

Theorem 4. *Suppose* [C.1], [C.2] *and* [C.3]. *Then for any* $g \in X'$ *and* $\Phi = {}^t(\varphi_1, \varphi_2, \cdots, \varphi_{2m}) \in \mathcal{D}(H^{2m}) \times \mathcal{D}(H^{2m-1}) \times \cdots \times \mathcal{D}(H)$, *the solution* $u(t)$ *of the initial value problem* (2)′, (3) *has the following asymptotic properties:*

(34) $\rho_\nu(H^{2m-j}\partial_t^{j-1}u(t) - H^{2m-j}(i\omega)^{j-1}v_{i\omega}^- e^{i\omega t}) \to 0$ $(j = 1, 2, \cdots, 2m)$

as $t \to \infty$, *for each* $\nu = 1, 2, \cdots$, *where* $v_{i\omega}^-$ *is the limit as* $\varepsilon \to +0$ *of the solution of* (32) *with* f *replaced by* $-g$ *in the sense of each* ρ_ν-*topology.*

4. Example. We consider strictly hyperbolic equations of the form

(35) $$\prod_{j=1}^{m}[\partial_t^2 + \alpha_j P(x, D)]u(x, t) = g(x)e^{i\omega t}, \quad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m,$$

in a domain G in $\boldsymbol{R}^n$ $(n \geq 3)$ exterior to a sufficiently smooth compact hypersurface ∂G, where

$$P(x, D) = -\sum_{j,k=1}^{n} D_j[a_{jk}(x)D_k] + c(x) \quad (D_j = \partial/\partial x_j)$$

We assume the followings:

[A.1] $a_{jk}(x)$ is real valued, $a_{jk}(x) = a_{kj}(x)$, and $\sum_{j,k}^{n} a_{jk}(x)\xi_j\xi_k \geq c|\xi|^2$ $(c > 0)$ for any $x \in \bar{G}$ and $\xi \in \boldsymbol{R}^n$. Further, $a_{jk}(x)$ is sufficiently smooth and $a_{jk}(x) - \delta_{jk}$ is of compact support.

[A.2] $c(x) \geq 0$. $c(x)$ is sufficiently smooth and

$$|c(x)|_{p,\theta} = \sup_{x \in G}(1 + |x|)^\theta \sum_{|\alpha| \leq p}|D_x^\alpha c(x)| < +\infty$$

for $p = [n/2] - 1$ and $\theta = (n + 1 + \delta)/2$ $(\delta > 0)$.

[A.3] $g(x) \in \mathcal{E}_{L^2,\text{loc}}^{[n/2]+1}(G)$[4] and

$$\|g(x)\|_{p,\theta} = \sum_{|\alpha| \leq p}\left\{\int_G (1 + |x|)^{2\theta}|D_x^\alpha g(x)|^2 dx\right\}^{1/2} < \infty$$

for $p = [n/2] + 1$ and $\theta = (n + \delta)/2$ $(\delta > 0)$.

We put the boundary conditions in one of the following form:

4) $\mathcal{E}_{L^2}^p(G)$ is the space of all functions such that $D_x^\alpha f(x) \in L^2(G)$, $|\alpha| \leq p$, with norm $(\sum_{|\alpha| \leq p}\|D_x^\alpha f\|_{L^2(G)}^2)^{1/2}$. $f \in \mathcal{E}_{L^2,\text{loc}}^p(G)$ if $\varphi f \in \mathcal{E}_{L^2}^p(G)$ for all C^∞-functions $\varphi(x)$ having compact supports in G.

(36) (Dirichlet type)

$$u|_{\partial G}=P(x,D)u|_{\partial G}=\cdots=P(x,D)^{m-1}u|_{\partial G}=0,$$

(37) (Neumann type)

$$\{\partial_n+\sigma(x)\}u|_{\partial G}=\{\partial_n+\sigma(x)\}P(x,D)u|_{\partial G}=\cdots$$
$$=\{\partial_n+\sigma(x)\}P(x,D)^{m-1}u|_{\partial G}=0,$$

where $\partial_n=\sum_{j,k}a_{jk}(x)\cos(x_j,\nu)D_k$, ν being the outer normal to ∂G at x, and $\sigma(x)\geq 0$ and is sufficiently smooth.

Now let $X=L^2(G)$ and L be the selfadjoint operator determined uniquely from $P(x,D)$ with domain

(Dirichlet case) $\mathcal{D}(L)=\mathcal{E}^2_{L^2}(G)\cap\mathcal{D}^1_{L^2}(G)$,[5)]

(Neumann case) $\mathcal{D}(L)=\{f\in\mathcal{E}^2_{L^2}(G)\,;\,(\partial_n+\sigma(x))u|_{\partial G}=0\}$.

Then, as is proved in Mochizuki [3], under the above assumptions,

$$(Lf,f)\geq 0 \qquad \text{for all } f\in\mathcal{D}(L)$$

and the spectrum of L is strongly absolutely continuous with respect to the Lebesgue measure. Further, if we put $Y=L^2_{\mathrm{loc}}(\bar{G})$ with semi-norms

$$\rho_\nu(f)=\left\{\int_{G_\nu}|f(x)|^2dx\right\}^{1/2},\quad G_\nu=\{x\in G\,;\,|x|\leq\nu\},$$

and X' being the set of functions $g(x)$ satisfying [A.3], then we know also in [3] that X and $H=L^{1/2}$ satisfy [C.1] and [C.2] in the previous section. Hence the assertions of Theorems 3 and 4 hold true for the solution of (35), (36) or (35), (37) if we give the initial data $\partial_t^{j-1}u|_{t=0}=\varphi_j(x)$ in $\mathcal{D}(H^{2m-j+1})$ $(j=1,2,\cdots,2m)$.

In conclusion, we remark that the function $v_{i\omega}(x)$ appearing in (24) satisfies the reduced equation

$$\prod_{j=1}^{m}[-\omega^2+\alpha_jP(x,D)]v_{i\omega}(x)=-g(x), \tag{38}$$

the boundary condition (36) or (37) and the radiation conditions formulated as follows:

$$v_{i\omega}(x)\in C^{2m-1}(G) \text{ and } \sum_{|\alpha|\leq 2m-1}|D_x^\alpha v_{i\omega}(x)|\leq \mathrm{const}(1+|x|)^{-(n-1)/2}; \tag{39}$$

$$\left(\sqrt{\alpha_j}\,\frac{d}{d|x|}+i\omega\right)\prod_k{}^{(j)}[-\omega^2+\alpha_kP(x,D)]v_{i\omega}(x)=0(|x|^{-\gamma-(n-1)/2}) \tag{40}$$

$$(j=1,2,\cdots,m),\quad \gamma=\min(1,2\delta/(1+\delta)),$$

where $\prod_k{}^{(j)}K_k=K_1K_2\cdots K_{j-1}K_{j+1}\cdots K_m$.

References

[1] Goldstein, J. A.: An asymptotic property of solutions of wave equations. Proc. Amer. Math. Soc., **20**, 359–363 (1969).

5) $\mathcal{D}^1_{L^2}(G)$ is the space obtained by the completion of $C_0^\infty(G)$ with respect to the $\mathcal{E}^1_{L^2}(G)$-norm, where $C_0^\infty(G)$ is the set of all C^∞-functions having compact supports in G.

[2] Mizohata, S.: Quelques problèmes au bord, du type mixte, pour des équations hyperboliques. Séminaire sur les équations aux dérivées partielles, Collège de France, 23–60 (1966–1967).

[3] Mochizuki, K.: The principle of limiting absorption connected with the exterior problem for second order elliptic operators and its application to the construction of eigenfunction expansions (to appear).

[4] Shinbrot, M.: Asymptotic behavior of solutions of abstract wave equations. Proc. Amer. Math. Soc., **19**, 1403–1406 (1968).