58. *Boundary Value Problems for Some Degenerate Elliptic Equations of Second Order with Dirichlet Condition*

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1. Introduction. Let Ω be a domain in R^n whose boundary is a smooth and compact hypersurface. We deal with the following differential operator defined in Ω:

$$A_\rho(x, D) = -\rho(r)\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial}{\partial x_k}\right) + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x) \tag{1.1}$$

where r denotes the distance from $x \in \bar{\Omega}$ to Γ, the boundary of Ω, and we assume that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geqq \delta|\xi|^2 \qquad \text{for any real } n\text{-vector } \xi \quad (a_{jk} = \bar{a}_{kj}), \tag{1.2}$$

and $\rho(t)$ $(t \in \bar{R}^1_+)$ satisfies

1) $\rho(t) \in C^0(\bar{R}^1_+) \cap C^2(R^1_+)$ and $0 \leqslant \rho(t)$ with $\rho(t) = 0$ only at $t = 0$
2) $\rho(t)^{-1}$ is integrable in $(0, s)$ for any $s \geqq 0$, and $\rho''(t) \leqq 0$ near $t = 0$
3) $|\rho'(t)| \leqq C_1 t^{\alpha-1}$ and $|\rho''(t)| \leqq C_2 t^{\alpha-2} (0 < \alpha < 1)$ near $t = 0$
4) $\int_0^a t^{2\alpha-2}\int_0^t \rho(s)^{-1} ds dt$ and $\int_0^a \rho'(t)\rho(t)^{-1}\int_0^t \rho(s)^{-1} ds dt$ are finite

for any $a > 0$ and if Ω is unbounded, we assume moreover

5) when $t \to \infty$, $0 < K \leqq \rho(t)$ and $\rho'(t)$, $\rho''(t)$ remain bounded.

If we take a function to be equal to t^α near $t = 0$ as $\rho(t)$, we can see easily that it satisfies the above conditions.

For the coefficients of $A_\rho(x, D)$, we assume that $a_{jk}(x)$ and $b_j(x)$ are all in $\mathscr{B}^1(\bar{\Omega})$, and $c(x)$ in $C^0(\Omega)$ with $|c(x)| \leqq M|\rho'(r)|\rho(r)^{-1}$ near Γ, and if Ω is unbounded, we assume that $c(x)$ remains bounded as $|x| \to \infty$.

Now let us introduce some Hilbert spaces in which we develop our arguments.

Definition 1.1. We say $u(x)$ belongs to $L^2(\Omega, \rho^{-1})$ if and only if

$$\|u\|^2_{0,\rho^{-1}} = \int_\Omega |u(x)|^2 \rho(r)^{-1} dx \tag{1.3}$$

is finite.

Definition 1.2. $u(x)$ is said to be in $H^m(\Omega, \rho)$, if and only if

$$\|u\|^2_{m,\rho} = \int_\Omega \Big(\rho(r)\sum_{|\alpha|=2} |D^\alpha u|^2 + |u|^2\Big) dx \tag{1.4}$$

is finite.

One of our main results is

Theorem 1.1. *Under the conditions stated above, the equation*

$$\begin{cases} A_\rho(x, D)u + \lambda u = f(x) \\ u|_\Gamma = 0 \end{cases} \tag{1.5}$$

admits a unique solution $u(x)$ *in* $H^2(\Omega, \rho)\cap \mathcal{D}^1_{L^2}(\Omega)$ *for any given* $f(x)$ *in* $L^2(\Omega, \rho^{-1})$, *if* $\lambda>0$ *is sufficiently large.*

For the adjoint equation, we have

Theorem 1.2. *The same result as Theorem* 1.1 *is valid for*

$$\begin{cases} A^*_\rho(x, D)v+\lambda v=g(x) \\ \qquad v|_\Gamma=0, \end{cases} \tag{1.6}$$

where $A^*_\rho(x, D)$ *stands for the formal adjoint operator of* $A_\rho(x, D)$ *with respect to the inner product of* $L^2(\Omega, \rho^{-1})$.

If Ω is bounded, we can see that the Fredholm alternative theorem holds.

In Section 4 we make mention of the application of our results to the mixed problems for the hyperbolic equations of second order.

2. Weak solution. We solve (1.5) by the so-called variational method. For this we prepare some lemmas.

Lemma 2.1. *If* $u(x)$ *belongs to* $H^1(\Omega, \rho)$, *then the trace of* $u(x)$ *to* Γ *exists and it holds for any positive* ε

$$\|u\|_\Gamma\leqq\varepsilon\|u\|_{1,\rho}+C(\varepsilon)\|u\|_0, \tag{2.1}$$

where $\|u\|_\Gamma$ *means the* $L^2(\Gamma)$ *norm of the trace of* $u(x)$.

Lemma 2.2. *Let* $u(x)$ *be in* $H^2(\Omega, \rho)$, *then the traces of* $D_ju(x)$ *exist and it holds for any positive* ε

$$\|D_ju\|_\Gamma\leqq\varepsilon\|u\|_{2,\rho}+C(\varepsilon)\|u\|_0, \qquad (j=1, \cdots, n). \tag{2.2}$$

Lemma 2.3. *Suppose* $u(x)\in H^2(\Omega, \rho)$, *then* $u(x)\in H^1(\Omega)$ *and it holds for any positive* ε

$$\|u\|_1\leqq\varepsilon\|u\|_{2,\rho}+C(\varepsilon)\|u\|_0. \tag{2.3}$$

Lemma 2.4. *Let* $u(x)$ *and* $v(x)$ *be in* $\mathcal{D}^1_{L^2}(\Omega)$, *then for any positive* ε

$$\int_\Omega \rho'(r)\rho(r)^{-2}|uv|\,dx\leqq\varepsilon(\|u\|_1^2+\|v\|_1^2)+C(\varepsilon)(\|u\|_0^2+\|v\|_0^2) \tag{2.4}$$

holds.

Lemma 2.5. *Let* $u(x)$ *and* $v(x)$ *be in* $\mathcal{D}^1_{L^2}(\Omega)$, *then for any first order differential operator* D, *we obtain with an arbitrary positive* ε

$$|(Du, v)_{\rho^{-1}}|\leqq\varepsilon(\|u\|_1^2+\|v\|_1^2)+C(\varepsilon)\|v\|_{0,\rho^{-1}}^2 \tag{2.5}$$

where $(\ ,\ )_{\rho^{-1}}$ *denotes the inner product of* $L^2(\Omega, \rho^{-1})$.

The final lemma is

Lemma 2.6. *If* $u(x)$ *is in* $\mathcal{D}^1_{L^2}(\Omega)$, *then we have*

$$\|u\|_0\leqq C\|u\|_{0,\rho^{-1}} \tag{2.6}$$

$$\|u\|_{0,\rho^{-1}}\leqq\varepsilon\|u\|_1+C(\varepsilon)\|u\|_0, \tag{2.7}$$

where ε *is an arbitrary positive number.*

Now let us define the weak solution of (1.5).

Definition 2.1. We say $u(x)$ in $\mathcal{D}^1_{L^2}(\Omega)$ a weak solution of (1.5), if $u(x)$ satisfies for all $v(x)$ in $\mathcal{D}^1_{L^2}(\Omega)$

$$\begin{aligned} B[u, v]= \sum_{j,k=1}^n\left(a_{jk}\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_j}\right)+\sum_{j=1}^n\left(b_j\frac{\partial u}{\partial x_j}, v\right)_{\rho^{-1}} \\ +(cu, v)_{\rho^{-1}}+\lambda(u, v)_{\rho^{-1}}=\langle f, \bar{v}\rangle. \end{aligned} \tag{2.8}$$

That $B[u, v]$ is well-defined follows from Lemma 2.4, Lemma 2.5 and Lemma 2.6, and using these lemmas again, we obtain

Proposition 2.1. *Let $u(x)$ and $v(x)$ be in $\mathcal{D}^1_{L^2}(\Omega)$, then it holds*

$$|B[u, v]| \leqq C\|u\|_1 \|v\|_1 \tag{2.9}$$

$$c\|u\|_1^2 \leqq \operatorname{Re} B[u, u], \tag{2.10}$$

if $\lambda > 0$ is large enough.

Thus by virtue of the well-known lemma of Lax-Milgram, we have

Theorem 2.1. *If $\lambda > 0$ is sufficiently large, then (1.5) has a unique weak solution for any $f(x)$ such that $\rho(r)^{-1} f(x)$ lies in $\mathcal{D}^1_{L^2}(\Omega)'$, especially for any $f(x)$ in $L^2(\Omega, \rho^{-1})$.*

3. Differentiability theorem. In this section we are concerned with the differentiability of the weak solution of (1.5). Since the question is local, we take $R^n_+ = \{(x, y)\,;\, x > 0 \text{ and } y \in R^{n-1}\}$ as Ω, and we may assume $\rho''(x) \leqq 0$ over R^1_+ without loss of generality.

Lemma 3.1. *Let $u(x, y) \in C_0^\infty(R^n_+)$ and $v(x, y) \in \mathcal{D}^1_{L^2}(R^n_+)$, then it holds*

$$(\rho(x) u_{xx}, v) = (\rho''(x) u, v) - (\rho(x) u_x, v_x) \tag{3.1}$$

Thus passing to limit, we obtain

Lemma 3.2. *If $u(x, y)$ is in $\mathcal{D}^1_{L^2}(R^n_+)$, then it follows*

$$\langle \rho(x) u_{xx}, \bar{u} \rangle = (\rho''(x) u, u) - (\rho(x) u_x, u_x). \tag{3.2}$$

Lemma 3.3. *Let $u(x, y)$ be in $\mathcal{D}^1_{L^2}(R^n_+)$, then we have with any $\varepsilon > 0$*

$$\|u\|_{0, \rho^{-1}} \leqq \varepsilon \|u\|_{1, \rho} + C(\varepsilon) \|u\|_0. \tag{3.3}$$

Lemma 3.4. *Let $u(x,y)$ be in $\mathcal{D}^1_{L^2}(R^n_+)$, then it holds*

$$|(\rho'(x) \rho(x)^{-1} u, u)|,\ \ |(Du, u)| \leqq \varepsilon \|u\|_{1,\rho}^2 + C(\varepsilon) \|u\|_0^2, \tag{3.4}$$

where ε is an arbitrary positive number and D stands for an arbitrary first order differential operator.

Let us denote by Σ_δ the hemi-sphere of radius δ with its centre the origin: $\Sigma_\delta = \{(x, y)\,;\, x^2 + |y|^2 < \delta^2 \text{ and } x > 0\}$.

Lemma 3.5 (Poincaré). *Let $u(x, y)$ be in $\mathcal{D}^1_{L^2}(R^n_+) \cap \mathcal{E}'(\Sigma_\delta)$, then it holds*

$$\|u\|_{0, \rho^{-1}}^2 \leqq C(\delta) \left\{ \sum_{j=1}^{n-1} \int_{R^n_+} \rho(x) \left| \frac{\partial u}{\partial y_j} \right|^2 dx dy + \int_{R^n_+} \rho(x) \left| \frac{\partial u}{\partial x} \right|^2 dx dy \right\} \tag{3.5}$$

where $C(\delta) \to 0$ as $\delta \to 0$.

Take δ sufficiently small, then we may assume that our concerning operator is of the form in Σ_δ

$$A_\rho = -\rho(x) \left\{ \frac{\partial}{\partial x} \left(\tilde{a}_{00} \frac{\partial}{\partial x} \right) + \sum_{j,k=1}^{n-1} \frac{\partial}{\partial y_j} \left(\tilde{a}_{jk} \frac{\partial}{\partial y_k} \right) \right\} + \text{first order operator}, \tag{3.6}$$

after a suitable local transformation of independent variables, and we may assume, by virtue of (1.2),

$$\sum_{j,k=1}^{n-1} \tilde{a}_{jk} \eta_j \eta_k \geqq c|\eta|^2 \text{ for any } \eta \in R^{n-1} \quad (\tilde{a}_{00} \geqq d > 0,\ c > 0 \text{ and } \tilde{a}_{jk} = \bar{\tilde{a}}_{kj}). \tag{3.7}$$

The following lemma is obvious.

Lemma 3.6. *For any $u(x,y)$ in $\mathcal{D}^1_{L^2}(R^n_+)$, it follows*

$$(3.8)\qquad -\left\langle \rho(x)\sum_{j,k=1}^{n-1}\frac{\partial}{\partial y_j}\left(a_{jk}\frac{\partial u}{\partial y_k}\right),\bar{u}\right\rangle \geqq c\int_{R^n_+}\rho(x)\sum_{j=1}^{n-1}\left|\frac{\partial u}{\partial y_j}\right|^2 dxdy.$$

Let us denote

$$B_\delta=\{(x,y)\,;\,x^2+|y|^2<\delta^2\}$$

and take an arbitrary real-valued function $\beta(x,y)$ belonging to $C_0^\infty(B_\delta)$.

Now let $u(x,y)$ be a weak solution of (1.5) with $\Omega=R^n_+$ and let $f(x,y)$ be in $L^2(R^n_+,\rho^{-1})$, then we see

$$(3.9)\qquad A_\rho u=f-\lambda u$$

as a distribution, and multiplying $\beta(x,y)$ to both sides we obtain

$$(3.10)\qquad A_\rho(\beta u)=\beta(f-\lambda u)-[\beta,A_\rho]u.$$

Lemma 3.7. *For any $u(x,y)$ in $\mathcal{D}^1_{L^2}(R^n_+)$, we have $[\beta,A_\rho]u \in L^2(R^n_+,\rho^{-1})$.*

Put $\beta u=v$ and $\beta(f-\lambda u)-[\beta,A_\rho]u=g$, then by Lemma 3.7 we have

$$(3.11)\qquad A_\rho v=g$$

where $v\in\mathcal{D}^1_{L^2}(R^n_+)\cap\mathcal{E}'(\Sigma_\delta)$ and $g\in L^2(\Sigma_\delta,\rho^{-1})$.

We denote by $H^m_0(\Sigma_\delta,\rho)$ the completion of $C_0^\infty(\Sigma_\delta)$ in $H^m(\Sigma_\delta,\rho)$ and denote by $H_0^{-m}(\Sigma_\delta,\rho)$ its dual space, which is a space of distribution.

Lemma 3.8. *If $u(x,y)$ is in $\mathcal{D}^1_{L^2}(\Sigma_\delta)$, then $A_\rho u$ is in $H_0^{-1}(\Sigma_\delta,\rho)$.*

The following proposition is essential in this section.

Proposition 3.1. *If δ is sufficiently small, then for any $u(x,y)$ in $\mathcal{D}^1_{L^2}(\Sigma_\delta)$ we get*

$$(3.12)\qquad \|A_\rho u\|_{-1,\rho}\geqq c\|u\|_{1,\rho},$$

where $\|A_\rho u\|_{-1,\rho}$ denotes the $H_0^{-1}(\Sigma_\delta,\rho)$ norm of $A_\rho u$.

Proof. By Lemma 3.2, Lemma 3.4 and Lemma 3.6, we have

$$(3.13)\qquad \mathrm{Re}\,\langle A_\rho u,\bar{u}\rangle\geqq c\|u\|^2_{1,\rho}-K\|u\|^2_0,$$

and by virtue of Lemma 3.5, we obtain

$$(3.14)\qquad \mathrm{Re}\,\langle A_\rho u,\bar{u}\rangle\geqq c'\|u\|_{1,\rho}.$$

Hence with the aid of Lemma 3.8, we can complete the proof.

Lemma 3.9. *If $f(x,y)$ is in $L^2(R^n_+,\rho)$ then the difference quotients $h^{-1}(f(x,y_1,\cdots,y_{j-1},y_j+h,y_{j+1},\cdots,y_{n-1})-f(x,y))(1\leqslant j\leqslant n-1)$ converge to $\dfrac{\partial f}{\partial y_j}$ in $H_0^{-1}(R^n_+,\rho)$.*

Thus applying Proposition 3.1 to v in (3.11) and using Lemma 3.9, we applying Proposition 3.1 to v in (3.11) and using Lemma 3.9, we can prove

Theorem 3.1. *If $u(x,y)\in\mathcal{D}^1_{L^2}(R^n_+)$ satisfies $A_\rho u=f$ with $f\in L^2(R^n_+,\rho^{-1})$, then $u(x,y)$ belongs to $H^2(R^n_+,\rho)$.*

Corollary 3.1. *If $u(x)\in\mathcal{D}^1_{L^2}(\Omega)$ satisfies $A_\rho u=f$ with $f\in L^2(\Omega,\rho^{-1})$, then $u(x)$ belongs to $H^2(\Omega,\rho)$.*

4. Application to mixed problems for hyperbolic equations. In this section we state an application of the results obtained in the previous sections to the mixed problems for hyperbolic equations

$$(4.1)\qquad \begin{cases} u_{tt}=A_\rho u+f(t,x) & \text{in } (0,T)\times\Omega \\ u(0,x)=u_0(x), & u_t(0,x)=u_1(x) \\ u(t,x)|_\Gamma=0 & \text{on } [0,T)\times\Gamma \end{cases}$$

We can show that (4.1) is well-posed in the following sense:

Theorem 4.1. *Let $b_j(x)$ $(j=1,\cdots,n)$ be all zero, then for any $f(t,x)\in\mathcal{E}_t^1(L^2(\Omega,\rho^{-1}))$ and for any $(u_0,u_1)\in H^2(\Omega,\rho)\cap\mathcal{D}_{L^2}^1(\Omega)\times\mathcal{D}_{L^2}^1(\Omega)$, there exists a unique solution $u(t,x)$ of (4.1) such that (u,u_t,u_{tt}) is continuous in $H^2(\Omega,\rho)\times H^1(\Omega)\times L^2(\Omega,\rho^{-1})$, and the energy estimate*

$$(4.2)\qquad \begin{aligned} \|u(t)\|_{2,\rho}+\|u_t(t)\|_1+\|u_{tt}(t)\|_{0,\rho^{-1}} &\leqq C(T)(\|u_0\|_{2,\rho}+\|u_1\|_1+\|f(0)\|_{0,\rho^{-1}} \\ &\quad +\int_0^t(\|f(s)\|_{0,\rho^{-1}}+\|f'(s)\|_{0,\rho^{-1}})ds) \end{aligned}$$

holds for any $t\in[0,T]$.

The more detailed exposition including the related topics will be published elsewhere.

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