100. *On a Ranked Vector Space*

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We show in this paper some relations between a ranked vector space and a linear topological space.

We suppose that a ranked vector space E satisfies the following conditions:

(M_1) Let E be a ranked vector space, a sequence, $\{u_n(x)\}$ any fundamental sequence of neighborhoods of an arbitrary point $x \in E$, and $v(x)$ any neighborhood of x (we denote this fact by $v(x) \in \mathfrak{V}(x)$), then there is a member $u_m(x)$ in $\{u_n(x)\}$ such that $u_m(x) \subset v(x)$.

Proposition 1. *Let E_1 and E_2 be two ranked vector spaces, and suppose that E_2 satisfies Condition* (M_1). *Let $f: E_1 \to E_2$ be continuous at a point $x \in E_1$, then for every neighborhood $v\{f(x)\}$ of the point $f(x) \in E_2$ there is a neighborhood $u(x)$ of the point $x \in E_1$ such that $f\{u(x)\} \subset v\{f(x)\}$.*

Proof. In order to show this, we proceed indirectly: i.e., assume that there is a neighborhood $v\{f(x)\}$ of the point $f(x)$ such that for any neighborhood $u(x)$ of x

$$f\{u(x)\} \not\subset v\{f(x)\}.$$

Let $\{u_n(x)\}$ be a fundamental sequence of neighborhoods of the point $x \in E_1$; i.e.,

$$u_0(x) \supset u_1(x) \supset u_2(x) \supset \cdots \supset u_n(x) \supset \cdots$$

and there is a sequence $\{\alpha_n\}$ of non-negative integers such that

$$\alpha_0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n \leqq \cdots$$

where $\sup\{\alpha_n\} = \infty$, and for each n, $u_n(x) \in \mathfrak{V}_{\alpha_n}$. By assumption we have that for any n

$$f\{u_n(x)\} \not\subset v\{f(x)\},$$

i.e., for each n there is an element x_n in $u_n(x)$ such that $f(x_n) \notin v\{f(x)\}$. Hence, it follows from the definition of convergence that $\{\lim x_n\} \ni x$ and $f(x_n) \notin v\{f(x)\}$ for every n. Since $f: E_1 \to E_2$ is continuous at x, by the definition of continuity it follows that

$$\{\lim f(x_n)\} \ni f(x).$$

Hence there is a fundamental sequence $\{v_n\{f(x)\}\}$ such that

$$v_0\{f(x)\} \supset v_1\{f(x)\} \supset v_2\{f(x)\} \supset \cdots \supset v_n\{f(x)\} \supset \cdots$$

$$\beta_0 \leqq \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n \leqq \cdots$$

$$\sup\{\beta_n\} = \infty \text{ and for every } n$$

$$v_n\{f(x)\} \in \mathfrak{V}_{\beta_n}, \qquad f(x_n) \in v_n\{f(x)\}.$$

For this $\{v_n\{f(x)\}\}$ it follows from Condition (M_1) that there is an integer m such that

$$v_m\{f(x)\} \subset v\{f(x)\}.$$
$$\therefore \quad f(x_m) \in v\{f(x)\}.$$

This contradicts that $f(x_m) \notin v\{f(x)\}$.

Proposition 2. *Let E be a ranked vector space satisfying Condition* (M_1), *then* $\mathfrak{V}(0)$ *has the following properties*:

(1) *for U and V in $\mathfrak{V}(0)$ there is a W in $\mathfrak{V}(0)$ such that $W \subset U \cap V$;*

(2) *for each V in $\mathfrak{V}(0)$ there is a member U of $\mathfrak{V}(0)$ such that $U+U \subset V$;*

(3) *for each U in $\mathfrak{V}(0)$ there is a member V of $\mathfrak{V}(0)$ such that $\lambda V \subset U$ for each scalar λ with $|\lambda| \leqq 1$; and*

(4) *for x in E and U in $\mathfrak{V}(0)$ there is a scalar $\lambda(\neq 0)$ such that $\lambda x \in U$.*

Proof. (1) and (3) are obvious. (2) If E is a ranked vector space, then $E \times E$ is also a ranked vector space. Since $f: E \times E \to E$ defined by $f\{(x, y)\} = x + y$ is continuous, it follows from Proposition 1 that for every $V \in \mathfrak{V}(0)$ there is a neighborhood $U \times U \in \mathfrak{V}_{E \times E}(0, 0)$ such that

$$f\{U \times U\} = U + U \subset V.$$

(4) Since the scalar multiplication $g: K \times E \to E$ defined by $g\{(\lambda, x)\} = \lambda x$ is continuous, for V in $\mathfrak{V}(0)$ there is a neighborhood $I \times u(x) \in \mathfrak{V}_{k \times E}\{(0, x)\}$ such that

$$g\{I \times u(x)\} \subset V.$$

Thus there is a $\lambda(\neq 0)$ with $\lambda x \in V$.

Proposition 2 shows that if a ranked vector space E satisfies Condition (M_1), $\mathfrak{V}(0)$ is a local base for a vector topology.

We now suppose that $\mathfrak{V}(0) = \{V; V \in \mathfrak{V}_n(0), n = 0, 1, 2, \cdots\}$ satisfies the following conditions:

(K_1) $\mathfrak{V}_0(0) \supset \mathfrak{V}_1(0) \supset \mathfrak{V}_2(0) \supset \cdots \supset \mathfrak{V}_n(0) \supset \cdots$;

(K_2) for each $\mathfrak{V}_n(0)$ there is a member U_n in $\mathfrak{V}_n(0)$ and an integer S_n such that every member V of $\mathfrak{V}_{s_n}(0)$ is contained in U_n.

From (K_2) we can consider a sequence

$$\mathfrak{U} = \{U_n; n = 0, 1, 2, \cdots\}$$

and we will show some properties of $\mathfrak{U}$.

Proposition 3. *Let E be a ranked vector space satisfying Conditions* (K_1), (K_2), *and $\{V_n\}$ a fundamental sequence of neighborhoods of zero, then for every $U \in \mathfrak{U}$ there is a member V_m in $\{V_n\}$ such that $V_m \subset U$.*

Proof. We may consider $U = U_l \in \mathfrak{U}$. From (K_2) it follows that there is an positive integer S_l such that for every V in $\mathfrak{V}_{s_l}$ $V \subset U_l$.

Since $\{V_n\}$ is a fundamental sequence of neighborhoods of zero, there is a sequence $\{\alpha_n\}$ of non-negative integers such that $\alpha_n \leqq \alpha_{n+1}$, for $n=0,1,2,\cdots$, $\sup\{\alpha_n\}=\infty$, and $V_n \in \mathfrak{B}_{\alpha_n}$ for $n=0,1,2,\cdots$. For S_l, it follows using $\sup\{\alpha_n\}=\infty$ that there is an α_m in $\{\alpha_n\}$ such that $\alpha_m \geqq S_l$. Thus by (K_1), (K_2) we have

$$V_m \in \mathfrak{B}_{\alpha_m} \subset \mathfrak{B}_{s_l},$$
$$\therefore \quad V_m \subset U_l = U.$$

Proposition 4. *Let E be a ranked vector space satisfying Conditions (K_1), (K_2) and $\{A_n\}$ a sequence of subsets of E such that every sequence $\{x_n\}$ with $x_n \in A_n (n=0,1,2,\cdots)$ converges to zero, then there is a subsequence $\{U_{t_i}\}$ of $\mathfrak{U}=\{U_n; n=0,1,2,\cdots\}$ such that*

$$A_i \subset U_{t_i} \qquad \text{for } i=0,1,2,\cdots.$$

Proof. For U_1 in $\mathfrak{U}$, there is an positive integer n_1 such that $U_1 \supset A_{n_1}, A_{n_1+1}, A_{n_1+2}, \cdots$. In order to show this, we proceed indirectly: i.e., assume that for any n there is an positive integer $m_1(>n)$ with $U_1 \not\supset A_{m_1}$. Then we can select the following sequences:

$$A_{m_1}, A_{m_2}, A_{m_3}, \cdots, A_{m_i}, \cdots$$
$$(m_1 < m_2 < m_3 < \cdots < m_i < \cdots)$$
$$x_i \in A_{m_i}, \quad \text{and} \quad x_i \notin U_1 \quad \text{for } i=1,2,3,\cdots.$$

By assumption we have

$$\{\lim x_i\} \ni 0$$

i.e., there is a fundamental sequence $\{V_i\}$ such that $x_i \in V_i$ for $i=1,2,3,\cdots$.

It follows from Proposition 3 that there is an positive integer m such that $x_m \in U_1$. This contradicts that $x_i \notin U_1$ for $i=1,2,3,\cdots$.

By (K_2), there is an positive integer S_{12} for U_1 such that each $V \in \mathfrak{B}_{s_{12}}$ is contained in U_1, and hence $U_{s_{12}} \subset U_1$. Thus we can select a sequence $\{U_{1i}\}$ such that

$$U_1 \supset U_{S_{12}} \supset U_{S_{13}} \supset \cdots \supset U_{S_{1i}} \supset \cdots \qquad ②$$

For $U_{S_{12}}$ we can show as before there is an positive integer $n_2(>n_1)$ such that

$$U_{S_{12}} \supset A_{n_2}, A_{n_2+1}, A_{n_2+2}, \cdots, A_{n_2+i}, \cdots$$

Continuing this process we can select a subsequence $\{U_{t_i}\}$ of $\mathfrak{U}=\{U_n, n=0,1,2,\cdots\}$ such that

$$A_i \subset U_{t_i} \qquad \text{for } i=1,2,3,\cdots.$$

Proposition 5. *Let E be a ranked vector space satisfying Conditions (K_1), (K_2), then $\mathfrak{U}=\{U_n; n=0,1,2,\cdots\}$ has the following properties:*

(i) *for $U, V \in \mathfrak{U}$ there is a W in $\mathfrak{U}$ such that $W \subset U \cap V$;*

(ii) *for every $V \in \mathfrak{U}$ there is a member U in $\mathfrak{U}$ such that $U+U \subset V$;*

(iii) *for every $U \in \mathfrak{U}$ there is a member V in $\mathfrak{U}$ such that $\lambda V \subset U$ for each scalar λ with $|\lambda| \leqq 1$; and*

(iv) *for every* $x \in E$ *and for every* $U \in \mathfrak{U}$ *there is a scalar* $\lambda (\lambda \neq 0)$ *such that* $\lambda x \in U$.

Proof. (i) Let $U = U_l$ and $V = U_m$, then there are positive numbers S_l and S_m such that $V' \subset U_l$ for every $V' \in \mathfrak{B}_{S_l}$ and $V'' \subset U_m$ for every $V'' \in \mathfrak{B}_{S_m}$. Let $n = \max(S_l, S_m)$, then

$$U_n \subset U \cap V.$$

(ii) Let V be an arbitrary element of $\mathfrak{U} = \{U_n ; n = 0, 1, 2, \cdots\}$, then V is denoted by $V = U_l$. From ② there exists a fundamental sequence of neighborhoods of zero such that

$$U_1 \supset U_{S_{12}} \supset U_{S_{13}} \supset \cdots \supset U_{S_{1i}} \supset \cdots.$$

Thus we have

$$U_1 + U_1 \supset U_{S_{12}} + U_{S_{12}} \supset U_{S_{13}} + U_{S_{13}} \supset \cdots \supset U_{S_{1i}} + U_{S_{1i}} \supset \cdots$$

Let $\{y_i\}$ be any sequence such that

$$y_i \in U_{S_{1i}} + U_{S_{1i}} \qquad (i = 1, 2, 3, \cdots) \qquad (S_{11} = 1)$$

then

$$y_i = x_i + x'_i$$

where $x_i, x'_i \in U_{S_{1i}}$ for $i = 1, 2, 3, \cdots$. It follows from $x_i, x'_i \in U_{S_{1i}} (i = 1, 2, 3, \cdots)$ that

$$\{\lim x_n\} \ni 0 \quad \text{and} \quad \{\lim x'_n\} \ni 0$$

Since E is a ranked vector space,

$$\{\lim (x_i + x'_i)\} \ni 0$$

$$\therefore \quad \{\lim y_i\} \ni 0$$

By Proposition 4 we have that there is a subsequence $\{W_i\}$ of $\mathfrak{U}$ such that

$$U_{S_{1i}} + U_{S_{1i}} \subset W_i \qquad i = 1, 2, 3, \cdots.$$

Hence, using Proposition 3 for $V = U_l$ there is a member $U_{S_{1m}}$ of ② such that

$$U_{S_{1m}} + U_{S_{1m}} \subset V = U_l.$$

(iii) This is clear.

(iv) Since E is a ranked vector space, it follows that for every $x \in E$ and for every sequence $\{\lambda_n\}$ with $\lim_{n \to \infty} \lambda_n = 0$ ($\lambda_n \neq 0$, $n = 1, 2, 3, \cdots$) there is a fundamental sequence $\{V_n\}$ of neighborhoods of zero such that

$$\lambda_n x \in V_n \qquad \text{for } n = 1, 2, 3, \cdots.$$

Let U be any element of $\mathfrak{U} = \{U_n ; n = 0, 1, 2, \cdots\}$, then there is an integer m such that

$$V_m \subset U$$

$$\therefore \quad \lambda_m x \in U$$

If E is a ranked vector space satisfying Conditions (K_1), (K_2), we can introduce in E a vector topology which has $\mathfrak{U}$ as a local base, then E is a linear topological space and it has a countable local base, and hence it is pseudometrizable.

Example. Let $c(-\infty, \infty)$ be a set of all continuous complex functions defined on $(-\infty, \infty)$, then clearly it is a linear space. We define a sequence $\{\|f\|_n\}$ of semi-norms as follows:

$$\|f\|_n = \sup\{|f(x)| : |x| \leqq n\}$$

for $n=1, 2, 3, \cdots$.

We now define the neighborhhood $v(n; 0)$ in the following way:

$$v(n; 0) = \left\{f ; \|f\|_n < \frac{1}{n}\right\}$$

for $n=1, 2, 3, \cdots$ and $v(0, 0) = c(-\infty, \infty)$.

Then $v(n; 0)$ has the following properties:

(1) each $v(n; 0)$ contains zero, and it is circled;

(2) if $m \geqq n$, then $v(m; 0) \subset v(n; 0)$;

(3) conversely, if $v(m; 0) \subset v(n; 0)$, then $m \geqq n$; and

(4) for any $v(m; 0)$, $v(n; 0)$ in $\{v(n; 0)\}$ there is a member $v(l; 0)$ in $\{v(n; 0)\}$ such that $v(l; 0) \subset v(m; 0) \cap v(n; 0)$.

Let

$$\mathfrak{V}_0(0) = \{v(0; 0), v(1, 0), v(2; 0), \cdots, v(n; 0), \cdots\}$$
$$\mathfrak{V}_1(0) = \{\qquad v(1, 0), v(2; 0), \cdots, v(n; 0), \cdots\}$$
$$\cdots\cdots$$
$$\mathfrak{V}_i(0) = \{\qquad\qquad v(i; 0), \cdots, v(n; 0), \cdots\}$$
$$\cdots\cdots$$

Then $c(-\infty, \infty)$ is a ranked vector space and it satisfies Conditions (K_1), (K_2).

References

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